UVA CS 6316: Machine Learning

Lecture 17: Generative Bayes Classifiers

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Course Content Plan ➔ Six major sections of this course

- ~~Regression (supervised)~~ — Y is a continuous
- Classification (supervised) — Y is a discrete
- Unsupervised models — NO Y
- Learning theory — About f()
- Graphical models — About interactions among X1,… Xp
- Reinforcement Learning — Learn program to Interact with its environment

Three major sections for classification

- We can divide the large variety of classification approaches into roughly three major types

1. Discriminative

    directly estimate a decision rule/boundary

    e.g~~., support vector machine~~, decision tree, ~~logistic regression,~~

    e.g~~. neural networks (NN), deep NN~~

2. Generative:

    build a generative statistical model

    e.g., Bayesian networks, Naïve Bayes classifier

3~~. Instance based classifiers~~

    - Use observation directly (no models)

    ~~- e.g. K nearest neighbors~~

Today : Generative Bayes Classifiers

- ✓ Bayes Classifier (BC)
  - Generative Bayes Classifier
- ✓ Naïve Bayes Classifier
- ✓ **Gaussian Bayes Classifiers**
  - **Gaussian distribution**
  - **Naïve Gaussian BC**
  - **Not-naïve Gaussian BC → LDA, QDA**

Review: Bayes classifiers (BC)

- Treat each feature attribute and the class label as random variables.

- Testing: Given a sample **x** with attributes ($x_1, x_2, \dots, x_p$):
  - Goal is to predict its class *c.*
  - Specifically, we want to find the class that maximizes $p(c \mid x_1, x_2, \dots, x_p)$.

- Training: can we estimate $p(C_i \mid \mathbf{x}) = p(C_i \mid x_1, x_2, \dots, x_p)$ directly from data?

$$c_{MAP} = \underset{c_j \in C}{\operatorname{argmax}} P(c_j \mid x_1, x_2, \dots, x_p)$$

MAP Rule

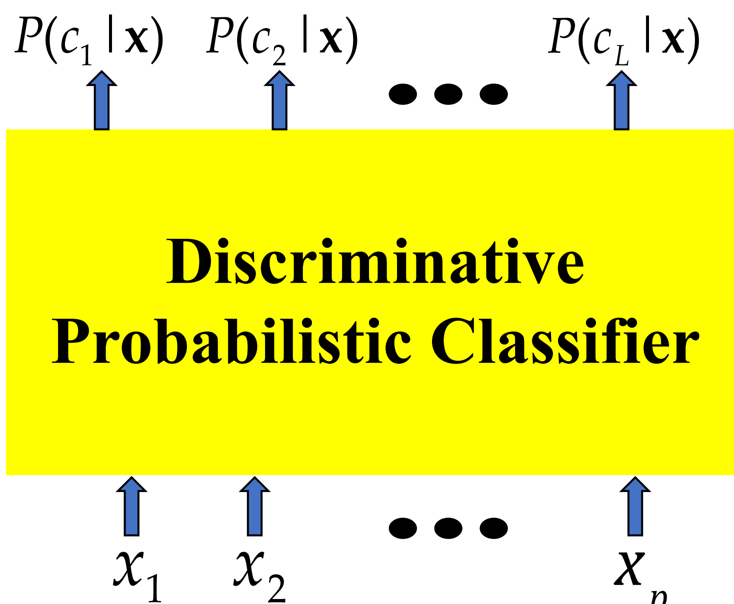
Two kinds of Bayes classifiers via MAP classification rule

- Establishing a probabilistic model for classification
  - **(1) Discriminative**
  - **(2) Generative**

Review: Discriminative BC

– **(1) Discriminative model**

$$\arg\max_{c \in C} P(c \,/\, \mathbf{X}), \quad C = \{c_1, \cdots, c_L\}$$



$$\mathbf{x} = (x_1, x_2, \cdots, x_p)$$

Softmax

$$p(\hat{y}_i | \vec{x}) = \frac{e^{z_i}}{\sum_{j=1}^{L} e^{z_j}}$$

multi-class

logistic regression

Adapt from Prof. Ke Chen NB slides

(2) Generative BC

$\Rightarrow p(c_i|\vec{x})$

$$P(\mathbf{X}|C),$$
$$C = c_1, \cdots, c_L, \mathbf{X} = (X_1, \cdots, X_p)$$

$P(\mathbf{x}|c_1)$ | $P(\mathbf{x}|c_2)$ | $P(\mathbf{x}|c_L)$

Generative Probabilistic Model for Class *1* | **Generative Probabilistic Model for Class *2*** | ... | **Generative Probabilistic Model for Class *L***

$x_1 \quad x_2 \quad \cdots \quad x_p$ | $x_1 \quad x_2 \quad \cdots \quad x_p$ | $x_1 \quad x_2 \quad \cdots \quad x_p$

$$\mathbf{x} = (x_1, x_2, \cdots, x_p)$$

Adapt from Prof. Ke Chen NB slides

Review Probability: If hard to directly estimate from data, most likely we can estimate

- 1. Joint probability
  - Use Chain Rule
- 2. Marginal probability
  - Use the total law of probability
- 3. Conditional probability
  - Use the Bayes Rule

$$p(c_i|\vec{x}) = \frac{p(\vec{x}|c_i)\, p(c_i)}{p(\vec{x})}$$

$$\Downarrow$$

$$c^* = \underset{c_i \sim \{c_1, c_2, \cdots, c_L\}}{\operatorname{argmax}} p(c_i|\vec{x})$$

Review : Bayes' Rule – for Generative Bayes Classifiers

$$\boxed{P(C|X) = \frac{P(X|C)P(C)}{P(X)}}$$

P(C$_1$|x), P(C$_2$|x), …, P(C$_L$|x)

P(C$_1$), P(C$_2$), …, P(C$_L$)

$$P(C_i|\mathbf{X}) = \frac{P(\mathbf{X}|C_i)P(C_i)}{P(\mathbf{X})}$$

Review : Bayes' Rule – for Generative Bayes Classifiers

$$\boxed{P(C \mid X) = \frac{P(X \mid C)P(C)}{P(X)}}$$

Prior

$P(C_1|x), P(C_2|x), \ldots, P(C_L|x)$

$P(C_1), P(C_2), \ldots, P(C_L)$

$$P(C_i \mid \mathbf{X}) = \frac{P(\mathbf{X} \mid C_i)P(C_i)}{P(\mathbf{X})}$$

Summary of Generative BC:

– Apply Bayes rule to get posterior probabilities

$$\underset{c_i}{\operatorname{argmax}} \left\{ P(C=c_i \mid \mathbf{X}=\mathbf{x}) \right\} = \frac{P(\mathbf{X}=\mathbf{x} \mid C=c_i)P(C=c_i)}{\cancel{P(\mathbf{X}=\mathbf{x})}}$$

$$\propto \underline{P(\mathbf{X}=\mathbf{x} \mid C=c_i)P(C=c_i)}$$

$$\text{for } i = 1, 2, \cdots, L$$

– Then apply the MAP rule

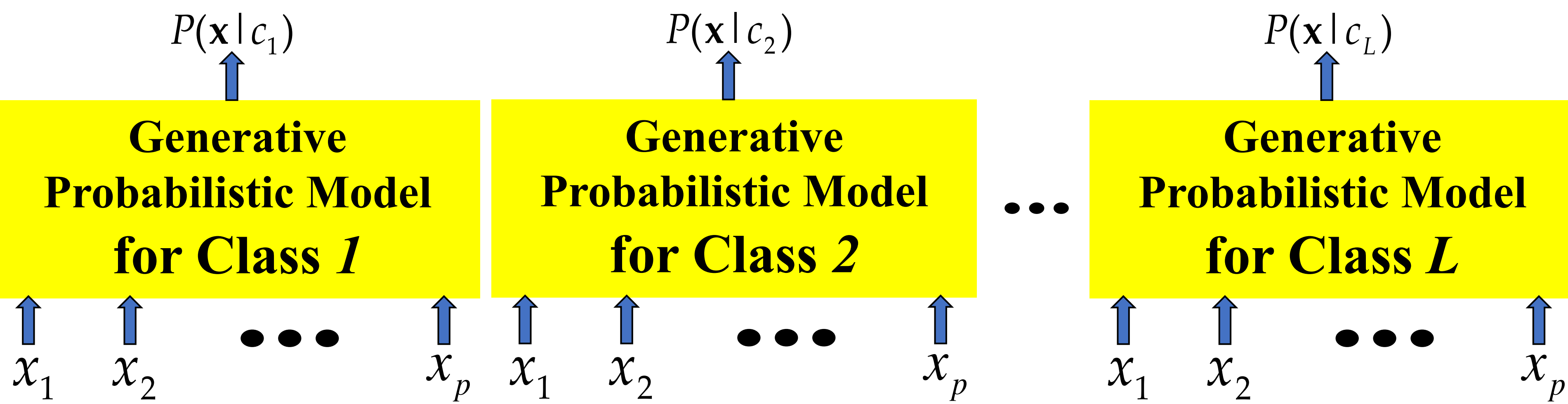
$$\begin{cases} p(\vec{x} \mid c_i) \\ p(c_i) \end{cases} \qquad c_i \in \{c_1, c_2, \cdots, c_L\}$$

Adapt from Prof. Ke Chen NB slides

Establishing a probabilistic model for classification through generative probabilistic models

[MAP rule]

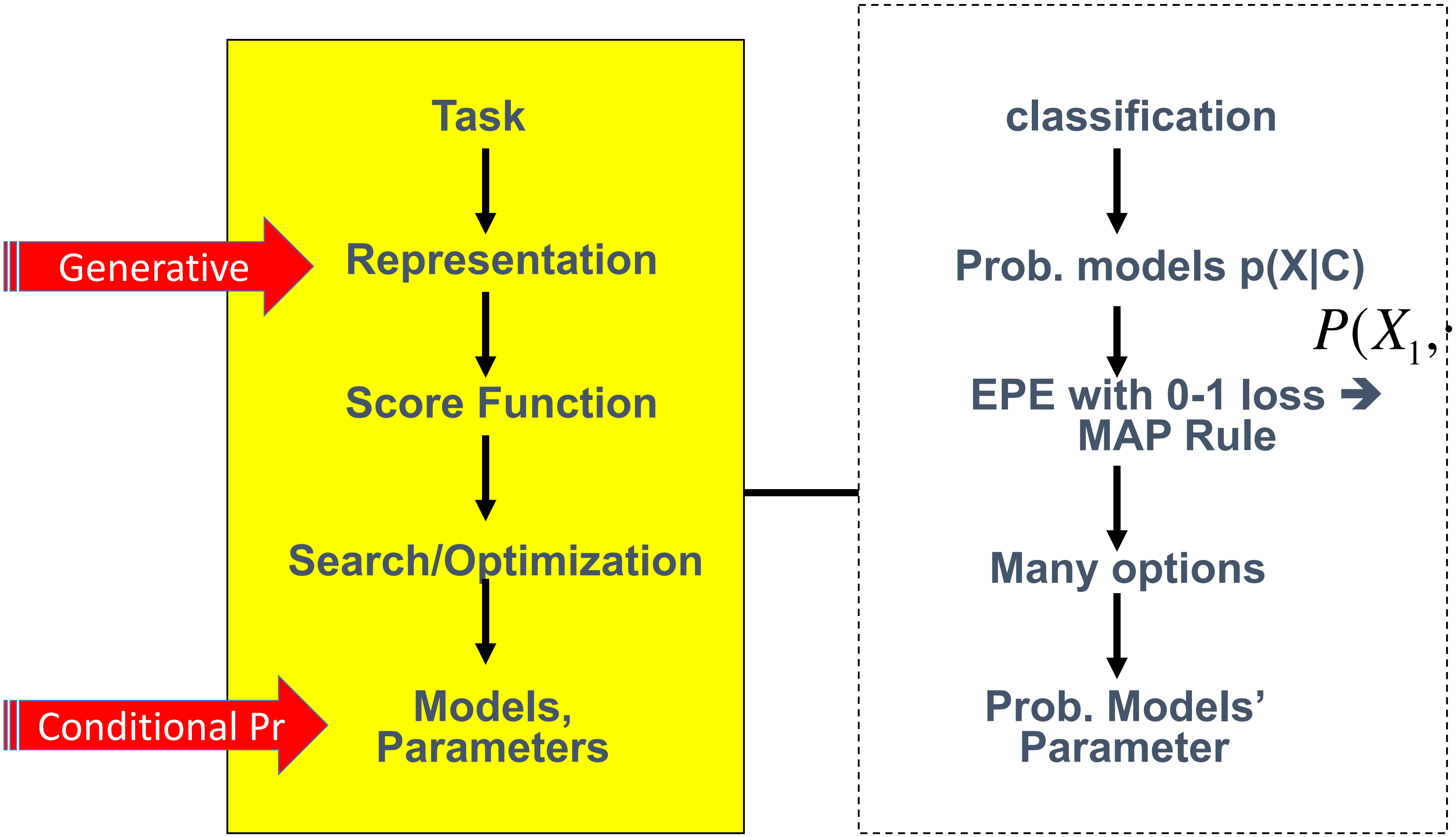
$$\underset{C_i}{\operatorname{argmax}} P(C_i \mid X) = \underset{C_i}{\operatorname{argmax}} P(X, C_i) = \underset{C_i}{\operatorname{argmax}} P(X \mid C_i) P(C_i)$$



$$\mathbf{x} = (x_1, x_2, \cdots, x_p)$$

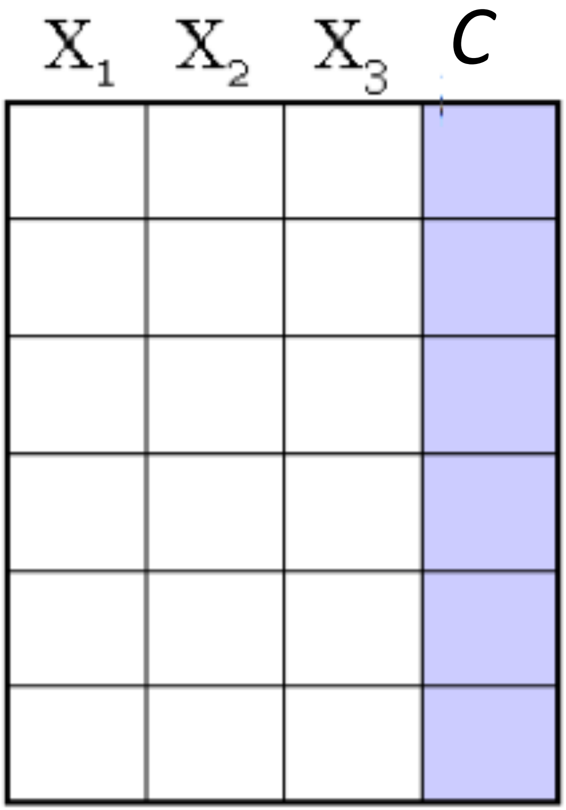
Adapt from Prof. Ke Chen NB slides

Generative Bayes Classifier



$$P(X_1, \cdots, X_p \mid C)$$

$$\underset{k}{\operatorname{argmax}} P(C\_k \mid X) = \underset{k}{\operatorname{argmax}} P(X, C) = \underset{k}{\operatorname{argmax}} P(X \mid C) P(C)$$



$$f : \boxed{X} \longrightarrow \boxed{C}$$

Output as Discrete Class Label $C_1, C_2, \ldots, C_L$

Discriminative

$$\underset{c \in C}{\arg\max}\, P(c \,/\, \mathbf{X}) \quad C = \{c_1, \cdots, c_L\}$$

Generative

$$\underset{c \in C}{\operatorname{argmax}}\, P(c \mid X) = \underset{c \in C}{\operatorname{argmax}}\, P(X, c) = \underset{c \in C}{\operatorname{argmax}}\, P(X \mid c) P(c)$$

this lecture!

X_1	X_2	X_3	C

An Example

- Example: Play Tennis

PlayTennis: training examples

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

X_1 X_2 X_3 X_4 C

X_1	X_2	X_3	X_4	C
S, O, R	H, M, C	H, N	W, S	C_1=Yes, C_2=No
3	× 3	× 2	× 2	× 2 ⇒ 72 parameters

$\Rightarrow P(X_1 X_2 X_3 X_4 \mid C), \; P(C)$ training

① $$\begin{cases} P(C=\text{Yes}) = \dfrac{N(\text{Yes})}{N(\text{train})} = \dfrac{9}{14} \\ P(C=\text{No}) = 1 - P(C=\text{Yes}) = \dfrac{5}{14} \end{cases}$$

② $$P(X_1=S, X_2=H, X_3=H, X_4=W \mid C=\text{No}) = \frac{N(\text{SHHWNo})}{N(\text{No})} = \frac{1}{5}$$

$$P(X_1=S, X_2=H, X_3=H, X_4=W \mid C=\text{Yes}) = \frac{0}{9} = 0$$

Learning Phase:

X_1	X_2	X_3	C

- Example: Play Tennis

X_1 X_2 X_3 X_4 C

PlayTennis: training examples

$K_2=3$

X_2: {Hot, Mild, Cool}

$X_3=${High, Normal}

$K_3=2$

$X_4=(W, S)$

$K_4=$

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

C: {Yes, No}

$L=2$

X_1: {sunny, overc, rain}

$K_1=3$

- Learning: maximum likelihood estimates
  - simply use the frequencies in the data

e.g. $p(\text{overcast}, \text{hot}, \text{high}, \text{weak} \mid \text{Yes}) = \frac{1}{9}$

Directly estimate from data

PlayTennis: training examples

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

e.g.

$p(\text{overcast}, \text{hot}, \text{high}, \text{weak} \mid \text{No})$

$= \frac{0}{9}$

Check L13-MLE Lecture for Why

$P(X_1, X_2, X_3, X_4 \mid Yes)$
$p(X_1, X_2, X_3, X_4 \mid No)$ → 36×2

X_1	X_2	X_3	C

PlayTennis: training examples

3

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

$3 \times 3 \times 2 \times 2 = 36$

$P(C = C_i)$
$\{Yes, No\}$

$P(C = Yes) = 9/14$

$P(C = No) = 5/14$

Generative Bayes Classifier:

e.g English Dictionary $X_1, \dots X_p$

$p \sim 30k$

$\Rightarrow C$ (up, down)

L

$2^p \times L$

- Learning Phase

$P(C_1), P(C_2), \dots, P(C_L)$

$p(C_1)$ $p(C_2)$

$P(\text{Play}=Yes) = 9/14 \quad P(\text{Play}=No) = 5/14$

$P(X_1,X_2, \dots, X_p|C_1), P(X_1,X_2, \dots, X_p|C_2)$

a look up table of cond. prob

Outlook (3 values)	Temperature (3 values)	Humidity (2 values)	Wind (2 values)	Play=*Yes*	Play=*No*
sunny	*hot*	*high*	*weak*	0/9	1/5
sunny	*hot*	*high*	*strong*	…/9	…/5
sunny	*hot*	*normal*	*weak*	…/9	…/5
sunny	*hot*	*normal*	*strong*	…/9	…/5
….	….	….	….	….	….
….	….	….	….	….	….
….	….	….	….	….	….
….	….	….	….	….	….

3*3*2*2 [conjunctions of attributes] * 2 [two classes]=72 parameters

Generative Bayes Classifier:

- **Testing Phase**
  - Given an unknown instance $\mathbf{X}'_{ts} = (a'_1, \cdots, a'_p)$
  - Look up tables to assign the label c^* to $\mathbf{X}_{ts}$ if

Last Page

$$\hat{P}(a'_1, \cdots a'_p \mid c^*)\hat{P}(c^*) > \hat{P}(a'_1, \cdots a'_p \mid c)\hat{P}(c),$$
$$c \neq c^*, \; c = c_1, \cdots, c_L$$

  - Given a new instance,

x'=(Outlook=*Sunny,* Temperature=*Cool,* Humidity=*High,* Wind=*Strong*)

$$\left\{ \begin{array}{l} p(x' \mid Yes)\, p(C=Yes) \\ p(x' \mid No)\, p(C=No) \end{array} \right\} \Rightarrow \arg\max_{C} \Rightarrow \text{predicted } C^*$$

$P(C=\text{Yes} \mid X_1, X_2, X_3, X_4)$

$P(C=\text{No} \mid X_1, X_2, X_3, X_4)$

$\rightarrow$ $P(C_1=\text{Yes}) = 9/14$

$P(C_2=\text{No}) = 5/14$

$\rightarrow$ $P(X_1, X_2, X_3, X_4 \mid C_i)$ parameter

$3 \times 3 \times 2 \times 2 \times 2 \Rightarrow 72$ from train

$\rightarrow$ $\underset{i=1,2}{\operatorname{argmax}} \; P(\bar{X}_{ts} \mid C_i) \, P(C_i)$ Generative BC

Today Recap: Generative Bayes Classifiers

- ✓ Bayes Classifier
  - Generative Bayes Classifier
- ✓ Naïve Bayes Classifier
- ✓ Gaussian Bayes Classifiers
  - Gaussian distribution
  - Naïve Gaussian BC
  - Not-naïve Gaussian BC → LDA, QDA

Naïve Bayes Classifier

- Bayes classification

$$\underset{c_j \in C}{\operatorname{argmax}} \underbrace{P(x_1, x_2, \ldots, x_p \mid c_j)}_{2 \times 2 \times \cdots \times 2 = 2^p \times L} P(c_j)$$

$\underline{c_j}$ L

Difficulty: learning the joint probability

- Naïve Bayes classification
  - Assumption that all input attributes are conditionally independent!

(when given C_i)

Naïve Bayes Classifier

$= p(x_1|c_j)\, p(x_2|c_j) \dots p(x_p|c_j)$

- Bayes classification

$$\underset{c_j \in C}{\operatorname{argmax}} P(x_1, x_2, \ldots, x_p | c_j) P(c_j)$$

Difficulty: learning the joint probability

- Naïve Bayes classification
  - Assumption that all input attributes are conditionally independent!

given C variable

Naïve Bayes Classifier

- Naïve Bayes classification
  - Assumption that all input attributes are conditionally independent!

[assumption]

$$P(X_1, X_2, \cdots, X_p | C) = P(X_1|C)P(X_2|C) \cdots P(X_p|C)$$

$$P(C_j | X_1, \ldots X_p) \propto P(X_1, X_2, \ldots X_p | C_j) P(C_j)$$

$$= \underline{P(X_1|C_j)} \, \underline{P(X_2|C_j)} \ldots \underline{P(X_p|C_j)} \, \underline{P(C_j)}$$

$3 \times L$		$P(X_1 \mid C)$
X_1 = Sunny	C=Yes	$\frac{N(S,Yes)}{N(Yes)} = \frac{2}{9}$
$X_1 = O$	C=Yes	
$X_1 = R$	C=Yes	
$X_1 = S$	C=No	$\frac{N(S,N)}{N(No)} = \frac{3}{5}$
O	C=No	
R	C=No	

$X_1 : 3 \times L$

$X_2 : 3 \times L$

$X_3 : 2 \times L$

$X_4 : 2 \times L$

L=2 $P(C=Ye)$	$P(C=N)$
9/14	5/14

No Naive

$72 + 2 = 74$ para

$2^{p} \times L$

$(3+3+2+2) \times \underbrace{L}_{=2} = 20$

$+2$

$\Rightarrow X_1, X_2, \ldots, X_p \Rightarrow \underbrace{2^{p} \times L}$

$p \sim 30k$

Naive Bayes Classifier

Task

↓

Representation

↓

Score Function

↓

Search/Optimization

↓

Models, Parameters

---

classification

↓

$P(X_1, X_2, \cdots, X_p | C) = P(X_1|C)P(X_2|C) \cdots P(X_p|C)$

Prob. models p(X|C)

↓

EPE with 0-1 loss ➔ MAP Rule

↓

Many options

↓

Prob. Models' Parameter $P(X_j|C_i), P(C_i)$

Estimate $P(X_j = x_{jk} | C = c_i)$ with examples in training;

$P(X_2|C_1), P(X_2|C_2)$

Outlook	Play=*Yes*	Play=*No*
Sunny		
Overcast		
Rain		

Temperature	Play=*Yes*	Play=*No*
Hot		
Mild		
Cool		

$P(X_4|C_1), P(X_4|C_2)$

Humidity	Play=*Yes*	Play=N *o*
High		
Normal		

Wind	Play=*Yes*	Play=*No*
Strong		
Weak		

P(Play=*Yes*) = ?? P(Play=*No*) = ??

$P(C_1), P(C_2), \ldots, P(C_L)$

Naïve Bayes Classifier

- Naïve Bayes classification
  - Assumption that all input attributes are conditionally independent!

$$P(X_1, X_2, \cdots, X_p | C) = P(X_1 | C) P(X_2 | C) \cdots P(X_p | C)$$

  - MAP classification rule: for a sample $\mathbf{x} = (x_1, x_2, \cdots, x_p)$

$$[P(x_1 | c^*) \cdots P(x_p | c^*)] P(c^*) > [P(x_1 | c) \cdots P(x_p | c)] P(c),$$

$$c \neq c^*, \ c = c_1, \cdots, c_L$$

Naïve Bayes Classifier

- Naïve Bayes classification
  - Assumption that all input attributes are conditionally independent!

$$P(X_1, X_2, \cdots, X_p \mid C) = P(X_1 \mid C) P(X_2 \mid C) \cdots P(X_p \mid C)$$

  - MAP classification rule: for a sample $\mathbf{x} = (x_1, x_2, \cdots, x_p)$

$$\underline{[P(x_1 \mid c^*) \cdots P(x_p \mid c^*)] P(c^*)} > [P(x_1 \mid c) \cdots P(x_p \mid c)] P(c),$$

$$c \neq c^*, \; c = c_1, \cdots, c_L$$

$\Rightarrow \underset{i=1,\cdots,L}{\arg\max} P(C_i) P(x_1 \mid C_i) P(x_2 \mid C_2) \cdots P(x_p \mid C_i)$

Naïve Bayes Classifier

- Naïve Bayes classification
  - Assumption that all input attributes are conditionally independent!

$$P(X_1, X_2, \cdots, X_p \mid C) = P(X_1 \mid C)P(X_2 \mid C) \cdots P(X_p \mid C)$$

  - MAP classification rule: for a sample $\mathbf{x} = (x_1, x_2, \cdots, x_p)$

$$[P(x_1 \mid c^*) \cdots P(x_p \mid c^*)]P(c^*) > [P(x_1 \mid c) \cdots P(x_p \mid c)]P(c),$$

$$c \neq c^*, \; c = c_1, \cdots, c_L$$

$\{j=1,2,\cdots,p\}$ $\{i=1,2,\cdots,L\}$

$P(X_j \mid C_i)$

Bernolli (p)
Binomial (k,p)
Multinomial
Gaussian

Naïve Bayes Classifier (for discrete input attributes) – training/ Learning phase

– Learning Phase: Given a training set S,

For each target value of c_i $(c_i = c_1, \cdots, c_L)$

$\hat{P}(C = c_i) \leftarrow$ estimate $P(C = c_i)$ with examples in $\mathbf{S}$;

Naïve Bayes Classifier (for discrete input attributes) – training/ Learning phase

– Learning Phase: Given a training set S,

For each target value of c_i $(c_i = c_1, \cdots, c_L)$

$\hat{P}(C = c_i) \leftarrow$ estimate $P(C = c_i)$ with examples in $\mathbf{S}$;

For every attribute value x_{jk} of each attribute X_j $(j = 1, \cdots, p;\ k = 1, \cdots, K_j)$

$\hat{P}(X_j = x_{jk} | C = c_i) \leftarrow$ estimate $P(X_j = x_{jk} | C = c_i)$ with examples in $\mathbf{S}$;

Output: conditional probability tables; for $X_j : K_j \times L$ elements

Naïve Bayes Classifier
(for discrete input attributes) - training

– Learning Phase: Given a training set S,

For each target value of c_i $(c_i = c_1, \cdots, c_L)$

$\hat{P}(C = c_i) \leftarrow$ estimate $P(C = c_i)$ with examples in **S**;

For every attribute value x_{jk} of each attribute X_j $(j = 1, \cdots, p;\ k = 1, \cdots, K_j)$

$\hat{P}(X_j = x_{jk} | C = c_i) \leftarrow$ estimate $P(X_j = x_{jk} | C = c_i)$ with examples in **S**;

$K_1, K_2, \ldots, K_p$

$\{X_1, X_2, \ldots, X_p\}$

$K_1 \times L +$
$K_2 \times L +$
$K_p \times L$

Naïve Bayes
(for discrete input attributes) - testing

– Test Phase: Given an unknown instance

Look up tables to assign the label c^* to **X'** if $\mathbf{X}' = (a_1', \cdots, a_p')$

$$[\hat{P}(a_1' | c^*) \cdots \hat{P}(a_p' | c^*)]\hat{P}(c^*) > [\hat{P}(a_1' | c) \cdots \hat{P}(a_p' | c)]\hat{P}(c),$$

$$c \neq c^*, \; c = c_1, \cdots, c_L$$

$P(X' | C_i) P(C_i)$

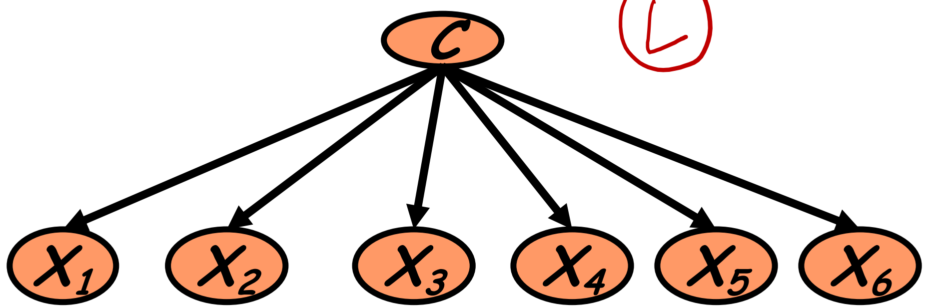
$= P(a_1' | C_i) P(a_2' | C_i) \cdots P(a_p' | C_i) P(C_i)$

$i = 1, 2, \cdots, L$

Learning (training) the NBC Model

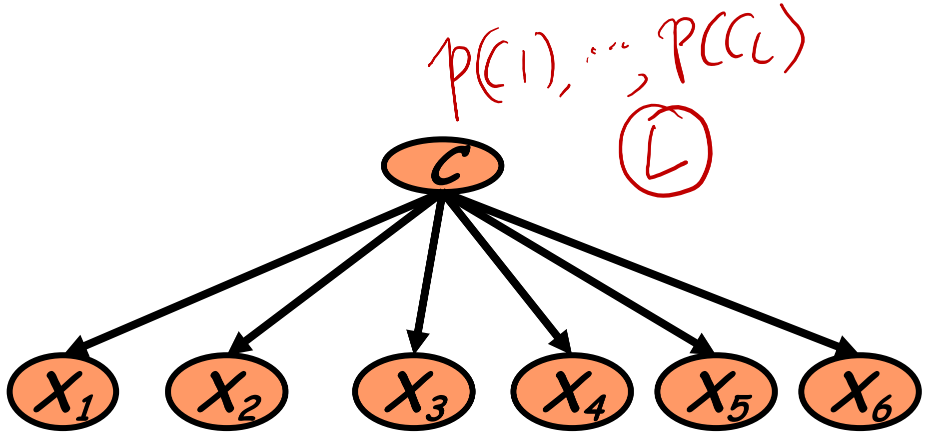
$p(C_1), \cdots, p(C_L)$

L



$p(X_k = \text{Hot} | C_i)$

Learning (training) the NBC Model



- maximum likelihood estimates:
  - simply use the frequencies in the data

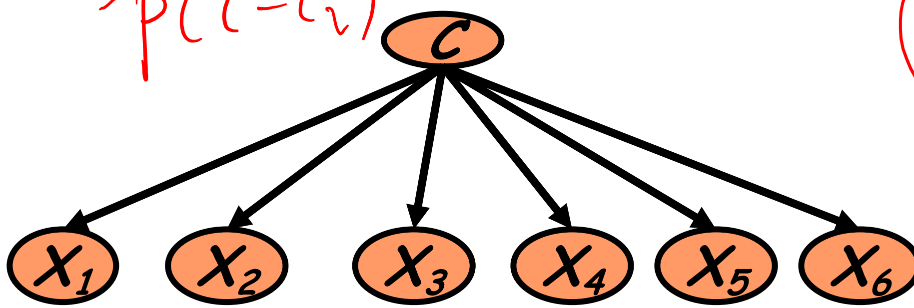
$$\hat{P}(c_j) = \frac{N(C = c_j)}{N}$$

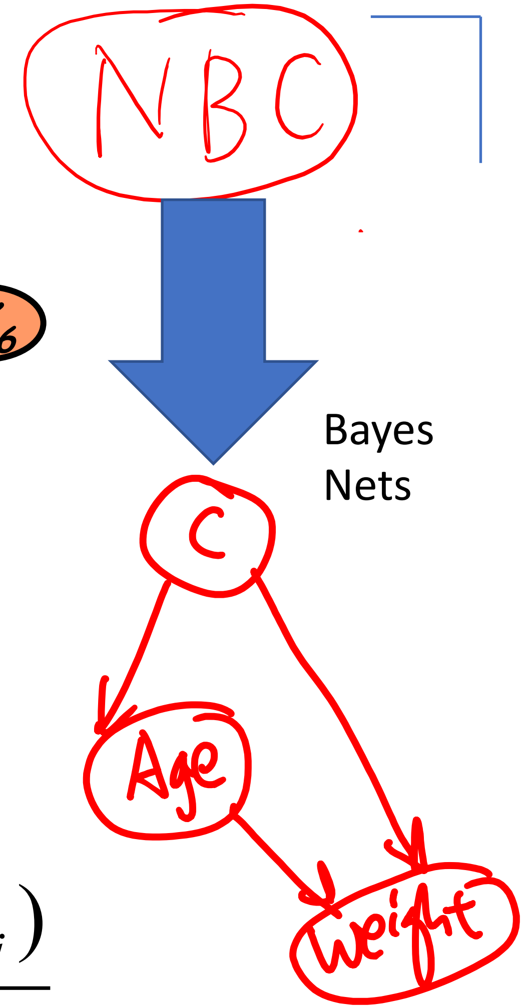
$$\hat{P}(x_i \mid c_j) = \frac{N(X_i = x_i, C = c_j)}{N(C = c_j)}$$

Learning (training) the NBC Model

$p(C=c_i)$

$p(X_j=x_i \mid c_i)$





Bayes Nets

- maximum likelihood estimates
  - simply use the frequencies in the data

$$\hat{P}(c_j) = \frac{N(C=c_j)}{N}$$

$$\hat{P}(x_i \mid c_j) = \frac{N(X_i = x_i, C=c_j)}{N(C=c_j)}$$

PlayTennis: training examples

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

$$p(X_1 = \text{Rain} \mid C = \text{Yes}) = \frac{3}{9}$$

$$p(X_1 = \text{Rain} \mid C = \text{No}) = \frac{2}{5}$$

Counting

- Learning Phase

Estimate $P(X_j = x_{jk} | C = c_i)$ with examples in training;

$P(X_2|C_1)$, $P(X_2|C_2)$

Outlook	Play=*Yes*	Play=*No*
Sunny	2/9	3/5
Overcast	4/9	0/5
Rain	3/9	2/5

Temperature	Play=*Yes*	Play=*No*
Hot	2/9	2/5
Mild	4/9	2/5
Cool	3/9	1/5

$P(X_4|C_1)$, $P(X_4|C_2)$

Humidity	Play=*Yes*	Play=*No*
High	3/9	4/5
Normal	6/9	1/5

Wind	Play=*Yes*	Play=*No*
Strong	3/9	3/5
Weak	6/9	2/5

3+3+2+2 [naïve assumption] * 2 [two classes]= 20 parameters

$P(\text{Play}=Yes) = 9/14 \quad P(\text{Play}=No) = 5/14$

$P(C_1)$, $P(C_2)$, …, $P(C_L)$

Counting

$P(X_1, X_2, X_3, X_4 | C_i)$

- Learning Phase

Estimate $P(X_j = x_{jk} | C = c_i)$ with examples in training;

$P(X_2|C_1)$, $P(X_2|C_2)$

X_1

2

2

3

3

Outlook	Play=*Yes*	Play=*No*
Sunny	2/9	3/5
Overcast	4/9	0/5
Rain	3/9	2/5

Temperature	Play=*Yes*	Play=*No*
Hot	2/9	2/5
Mild	4/9	2/5
Cool	3/9	1/5

2

$P(X_4|C_1)$, $P(X_4|C_2)$

2

2

2

Humidity	Play=*Yes*	Play=N*o*
High	3/9	4/5
Normal	6/9	1/5

Wind	Play=*Yes*	Play=*No*
Strong	3/9	3/5
Weak	6/9	2/5

$\ll 72$

3+3+2+2 [naïve assumption] * 2 [two classes]= 20 parameters

$P(\text{Play}=Yes) = 9/14 \quad P(\text{Play}=No) = 5/14$

$P(C_1)$, $P(C_2)$, …, $P(C_L)$

$P(C_i)$

Testing the NBC Model

look up

$$[\hat{P}(a_1'|c^*)\cdots\hat{P}(a_p'|c^*)]\hat{P}(c^*) > [\hat{P}(a_1'|c)\cdots\hat{P}(a_p'|c)]\hat{P}(c)$$

- Test Phase
  - Given a new instance,

    x'=(Outlook=*Sunny,* Temperature=*Cool,* Humidity=*High,* Wind=*Strong*)

Using the NBC Model

$$[\hat{P}(a_1'|c^*)\cdots\hat{P}(a_p'|c^*)]\hat{P}(c^*) > [\hat{P}(a_1'|c)\cdots\hat{P}(a_p'|c)]\hat{P}(c)$$

- Test Phase
  - Given a new instance,
    x'=(Outlook=*Sunny*, Temperature=*Cool*, Humidity=*High*, Wind=*Strong*)

$\rightarrow P(C_1)\,P(Sunny|C_1)\,P(Cool|C_1)\,P(High|C_1)\,P(Strong|C_1)$

$= \frac{9}{14} \times \frac{2}{9} \cdots =$

$\rightarrow P(C_2)\,P(Su|C_2)\,P(Co|C_2)\,P(hi|C_2)\,P(St|C_2)$

$= \frac{5}{14} \times \frac{3}{5} \times \cdots =$

Testing the NBC Model

$$[\hat{P}(a_1'|c^*)\cdots\hat{P}(a_p'|c^*)]\hat{P}(c^*) > [\hat{P}(a_1'|c)\cdots\hat{P}(a_p'|c)]\hat{P}(c)$$

- Test Phase
  - Given a new instance,
    x'=(Outlook=*Sunny*, Temperature=*Cool*, Humidity=*High*, Wind=*Strong*)
  - Look up in conditional-prob tables

P(Outlook=*Sunny*|Play=*Yes*) = 2/9	P(Outlook=*Sunny*|Play=*No*) = 3/5
P(Temperature=*Cool*|Play=*Yes*) = 3/9	P(Temperature=*Cool*|Play==*No*) = 1/5
P(Huminity=*High*|Play=*Yes*) = 3/9	P(Huminity=*High*|Play=*No*) = 4/5
P(Wind=*Strong*|Play=*Yes*) = 3/9	P(Wind=*Strong*|Play=*No*) = 3/5
P(Play=*Yes*) = 9/14	P(Play=*No*) = 5/14

$$[\hat{P}(a_1'|c^*)\cdots\hat{P}(a_p'|c^*)]\hat{P}(c^*) > [\hat{P}(a_1'|c)\cdots\hat{P}(a_p'|c)]\hat{P}(c)$$

- Test Phase
  - Given a new instance,

    x'=(Outlook=*Sunny*, Temperature=*Cool*, Humidity=*High*, Wind=*Strong*)
  - Look up in conditional-prob tables

P(Outlook=*Sunny*|Play=*Yes*) = 2/9	P(Outlook=*Sunny*|Play=*No*) = 3/5
P(Temperature=*Cool*|Play=*Yes*) = 3/9	P(Temperature=*Cool*|Play==*No*) = 1/5
P(Huminity=*High*|Play=*Yes*) = 3/9	P(Huminity=*High*|Play=*No*) = 4/5
P(Wind=*Strong*|Play=*Yes*) = 3/9	P(Wind=*Strong*|Play=*No*) = 3/5
P(Play=*Yes*) = 9/14	P(Play=*No*) = 5/14

  - MAP rule

    P(*Yes*|**x**'): [P(*Sunny*|Yes)P(*Cool*|*Yes*)P(*High*|Yes)P(*Strong*|*Yes*)]P(Play=*Yes*) = 0.0053

    P(*No*|**x**'): [P(*Sunny*|No) P(*Cool*|No)P(*High*|*No*)P(*Strong*|*No*)]P(Play=*No*) = 0.0206

Given the fact P(*Yes*|**x**') < P(*No*|**x**'), we label **x**' to be "*No*".

WHY ? Naïve Bayes Assumption

Not Naïve

Naïve

- $P(c_j)$
  - Can be estimated from the frequency of classes in the training examples.
- $P(x_1,x_2,\ldots,x_p/c_j)$
  - $O(|X_1|.\ |X_2|.\ |X_3|\ldots.\ |X_p|.|C|)$ parameters
  - Could only be estimated if a very, very large number of training examples was available.
- $P(x_k/c_j)$
  - $O([|X_1|+\ |X_2|+\ |X_3|\ldots.+\ |X_p|].|C|)$ parameters
  - Assume that the probability of observing the conjunction of attributes is equal to the product of the individual probabilities $P(x_i|c_j)$.

WHY ? Naïve Bayes Assumption

Assuming $|c| = L$

num of unique values

Not Naïve

- $P(c_j)$
  - Can be estimated from the frequency of classes in the training examples.

Assuming $|X_i| = 2$, $i = 1, 2, \cdots, p$

$\Rightarrow 2^p \times L$ Exp

- $P(x_1, x_2, \ldots, x_p / c_j)$
  - **$O(|X_1| \cdot |X_2| \cdot |X_3| \ldots |X_p| \cdot |C|)$ parameters**
  - Could only be estimated if a very, very large number of training examples was available.

$(2 + 2 + 2 + \cdots 2) \times L$

$= 2 \times p \times L$ Linear

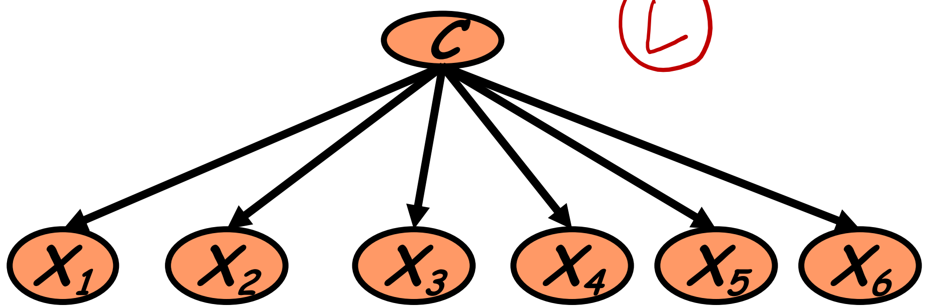
Naïve

- $P(x_k / c_j)$
  - **$O([|X_1| + |X_2| + |X_3| \ldots + |X_p|] \cdot |C|)$ parameters**
  - Assume that the probability of observing the conjunction of attributes is equal to the product of the individual probabilities $P(x_i | c_j)$.

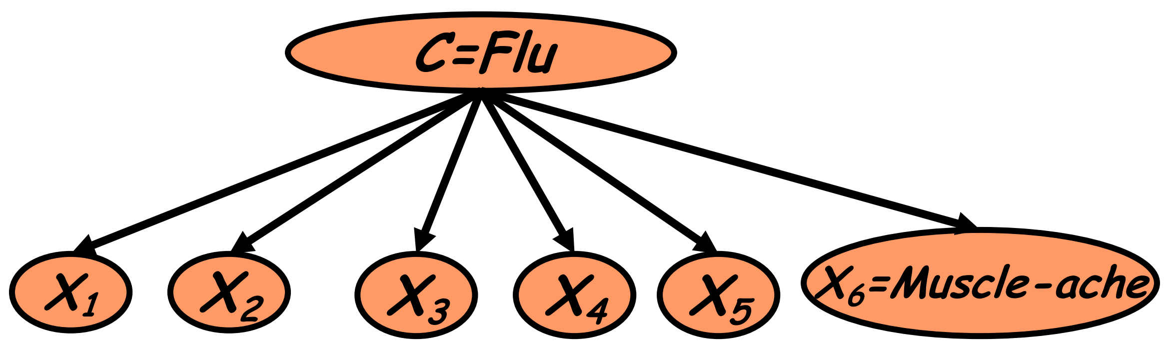
Challenges during Learning (training) the NBC Model

$P(C_1), \ldots, P(C_L)$

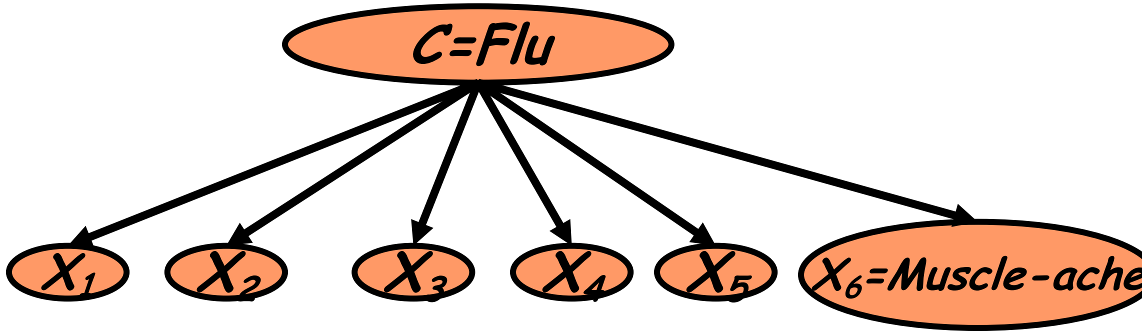
L



For instance:



For instance:



- What if we have seen no training cases where patient had no flu and muscle aches?

$$\hat{P}(X_6 = T | C = not\_flu) = \frac{N(X_6 = T, C = nf)}{N(C = nf)} = 0$$

$\overline{\text{muscle-ache=Yes/no}}$ $\overline{\text{flu/nf}}$

- Zero probabilities cannot be conditioned away, no matter the other evidence!

$$?? = \arg\max_c \hat{P}(c) \prod_i \hat{P}(x_i | c)$$

$$\delta_f = p(c=f|u)\, p(x_1|f)\, p(x_2|f)\, p(x_3|f)\, p(x_4|f)\, p(x_5|f)\, p(x_6|f)$$

$$\delta_{nf} = p(c=nf)\, p(x_1|nf)\, p(x_2|nf)\, p(x_3|nf)\, p(x_4|nf)\, p(x_5|nf)\, p(x_6|nf)$$

if any term gives 0,

$\Rightarrow \delta_{nf} = 0$

no matter other terms' value

Smoothing to Avoid Overfitting

Why necessary ??

$$\hat{P}(x_i \mid c_j) = \frac{N(X_i = x_i, C = c_j) + 1}{N(C = c_j) + k_i}$$

# of values of feature X_i

To make
sum_i (P(xi|Cj)=1

$|X_i| = k_i$

Adapt From Manning' textCat tutorial

Smoothing to Avoid Overfitting

$$\hat{P}(x_i \mid c_j) = \frac{N(X_i = x_i, C = c_j) + 1}{N(C = c_j) + k_i}$$

# of values of X_i

- Somewhat more subtle version

overall fraction in data where $X_i = x_{i,k}$

$$\hat{P}(x_{i,k} \mid c_j) = \frac{N(X_i = x_{i,k}, C = c_j) + m p_{i,k}}{N(C = c_j) + m}$$

$k \in \{1, 2, \ldots, k_i\}$

extent of "smoothing"

Summary:
Generative Bayes Classifier

Task: Classify a new instance X based on a tuple of attribute values $X = \langle X_1, X_2, \ldots, X_p \rangle$ into one of the classes

$$c_{MAP} = \underset{c_j \in C}{\operatorname{argmax}} P(c_j \mid x_1, x_2, \ldots, x_p)$$

$$= \underset{c_j \in C}{\operatorname{argmax}} \frac{P(x_1, x_2, \ldots, x_p \mid c_j) P(c_j)}{P(x_1, x_2, \ldots, x_p)}$$

$$= \underset{c_j \in C}{\operatorname{argmax}} \underbrace{P(x_1, x_2, \ldots, x_p \mid c_j)} \underbrace{P(c_j)}$$

$j = 1, 2, \ldots, L$

MAP = Maximum A Posteriori

Adapt From Carols' prob tutorial

NEXT: More Generative Bayes Classifiers

- ✓ Generative Bayes Classifier
- ✓ Naïve Bayes Classifier
- ✓ Gaussian Bayes Classifiers
  - Gaussian distribution
  - Naïve Gaussian BC
  - Not-naïve Gaussian BC → LDA, QDA
- ✓ Discriminative vs. Generative

$$\operatorname*{argmax}_k P(C\_k \mid X) = \operatorname*{argmax}_k P(X, C) = \operatorname*{argmax}_k P(X \mid C) P(C)$$

Generative Bayes Classifier

Task	classification
↓	↓
Representation	Prob. models p(X|C)
↓	↓
Score Function	EPE with 0-1 loss ➔ MAP Rule
↓	↓
Search/Optimization	Many options
↓	↓
Models, Parameters	Prob. Models' Parameter

$$P(X_1, \cdots, X_p \mid C)$$

Gaussian Naive

$$\hat{P}(X_j \mid C = c_k) = \frac{1}{\sqrt{2\pi}\sigma_{jk}} \exp\left(-\frac{(X_j - \mu_{jk})^2}{2\sigma_{jk}^2} \right)$$

Bernoulli Naïve

$$p(W_i = true \mid c_k) = p_{i,k}$$

Multinomial

$$P(W_1 = n_1, \ldots, W_v = n_v \mid c_k) = \frac{N!}{n_{1k}! n_{2k}! .. n_{vk}!} \theta_{1k}^{n_{1k}} \theta_{2k}^{n_{2k}} .. \theta_{vk}^{n_{vk}}$$

References

- Prof. Andrew Moore's review tutorial
- Prof. Ke Chen NB slides
- Prof. Carlos Guestrin recitation slides