UVA CS 4774: Machine Learning

Lecture 5: Linear Regression with Basis Functions Expansion

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Today : Multivariate (non-) Linear Regression with Basis Expansion

Data: X
↓
Task: y
↓
Representation: : x, f()
↓
Score Function: L()
↓
Search/Optimization : argmin()
↓
Models, Parameters

⇒

X: Tabular
↓
Regression: y continuous
↓
Y = Weighted linear sum of (X basis expansion)
↓
Sum of Squared Error (Least Squared)
↓
Normal Equation / GD / SGD
↓
Metrics, Implementation, Regression coefficients w, b

$$\hat{y} = \theta_0 + \sum_{j=1}^{m} \theta_j \varphi_j(x) = \varphi(x)^T \theta$$

Linear Regression with non-linear basis functions

- LR can deal with nonlinear relationships

$$\hat{y} = \theta^T \mathbf{x} \quad \Longrightarrow \quad \hat{y} = \theta_0 + \sum_{j=1}^{m} \theta_j \varphi_j(\mathbf{x}) = \theta^T \varphi(\mathbf{x})$$

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$$\vec{\theta} = [\theta_0, \theta_1, \cdots, \theta_m]^T$$

$$\vec{\Phi} = [1, \varphi_1(x), \cdots, \varphi_m(x)]^T$$

LR with non-linear basis functions

- Free to design basis functions (e.g., non-linear features:

Here $\varphi_j(x)$ are predefined basis functions (also $\varphi_0(x) = 1$)

- E.g.: polynomial regression with degree up-to two (d=2) :

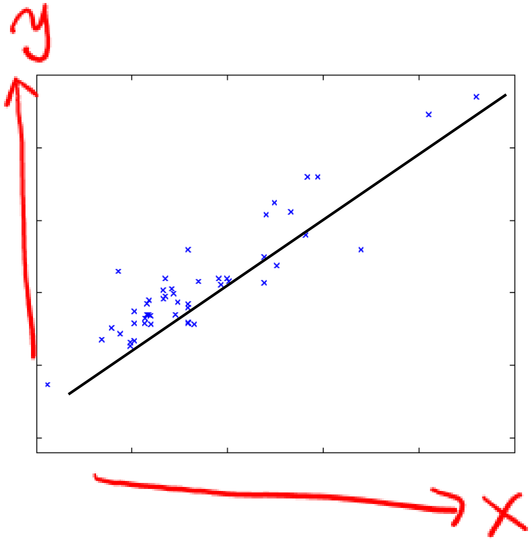
$$\varphi(x) := \left[1, x, x^2\right]^T$$

linear $\vec{x} : [1, x]^T$

Polynomial basis based Regression

e.g. (1) polynomial regression

$$\hat{y} = \theta^T \mathbf{x} \quad \Rightarrow \quad \hat{y} = \theta^T \varphi(\mathbf{x})$$



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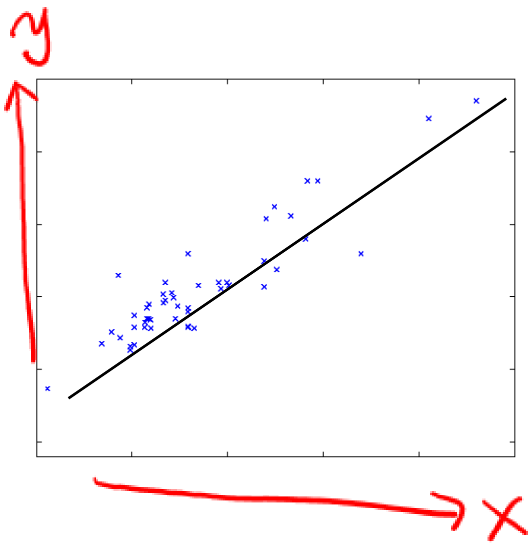
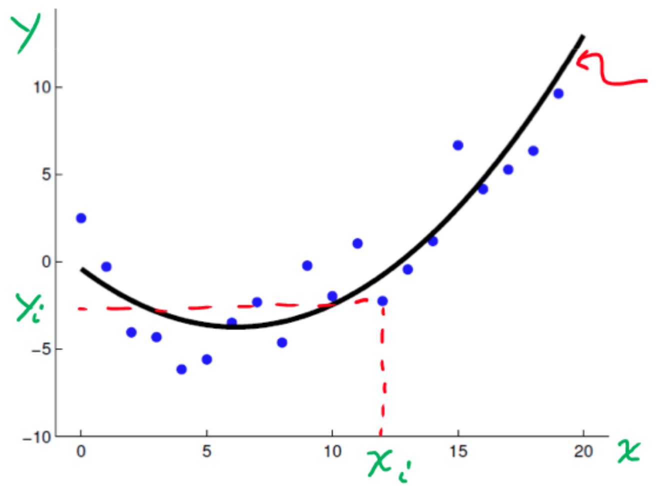
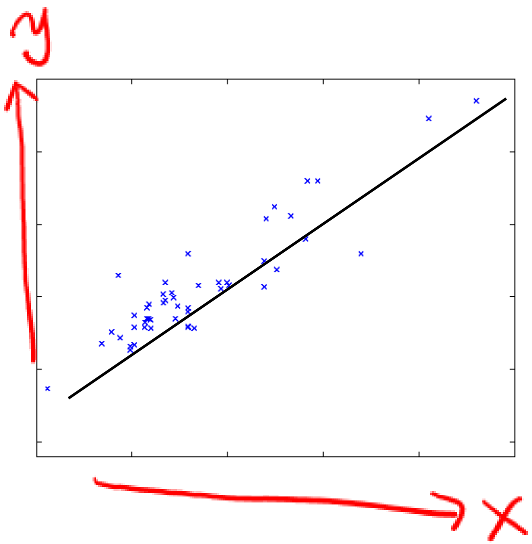




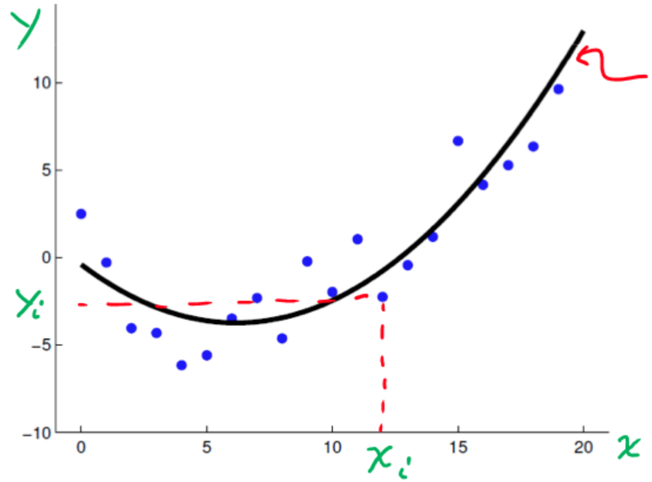
$= \theta_0 + x\theta_1 + x^2\theta_2$

$$\hat{y} = \theta^T \mathbf{x}$$



$$\theta^* = \left(X^T X\right)^{-1} X^T \vec{y}$$

$$\hat{y} = \theta^T \varphi(\mathbf{x})$$



$= \theta_0 + x\theta_1 + x^2\theta_2$

$$\theta^* = \left(\varphi^T \varphi\right)^{-1} \varphi^T \vec{y}$$

$$\varphi(x) := \left[1, x, x^2\right]^T$$

feature engineering

linear $\vec{x} : [1, x]^T$

$$\theta^* = \left(X^T X\right)^{-1} X^T \vec{y}$$

$(p+1)$

$X_{n \times p} =$ 1, 2, ⋮, n $\left[\quad\right]$

p=1

$$\varphi(x) := \left[\underset{1}{1}, \underset{2}{x}, \ldots, \underset{m}{x^2}\right]^T$$

$$\theta^* = \left(\underbrace{\underset{m\times n}{\varphi^T}\underset{n\times m}{\varphi}}\right)^{-1} \underbrace{\underset{m\times n}{\varphi^T}\underset{n\times 1}{\vec{y}}}_{}\ \ m\times 1$$

$\Phi(x) =$ 1, 2, 3, ⋮, n $\left[\quad\right]$ 1 . . . m

$\Downarrow$

$\vec{\theta} \in R^m$

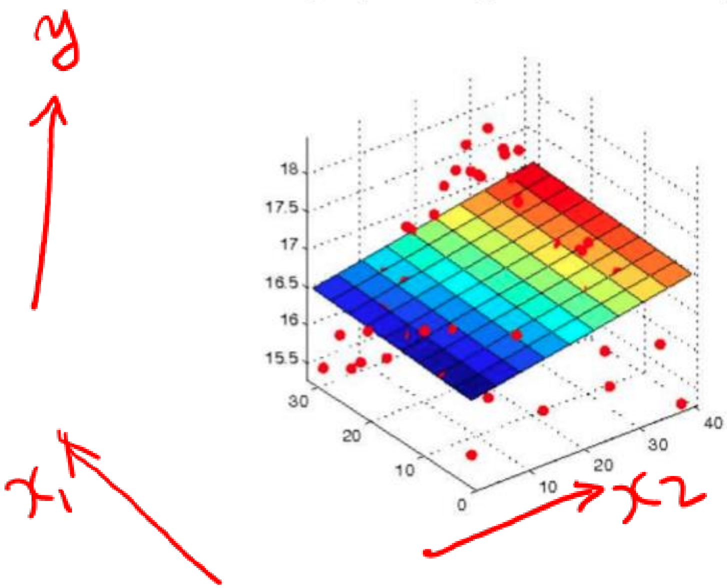
KEY: when the basis func are given, the problem of learning the parameters from data is still LR.

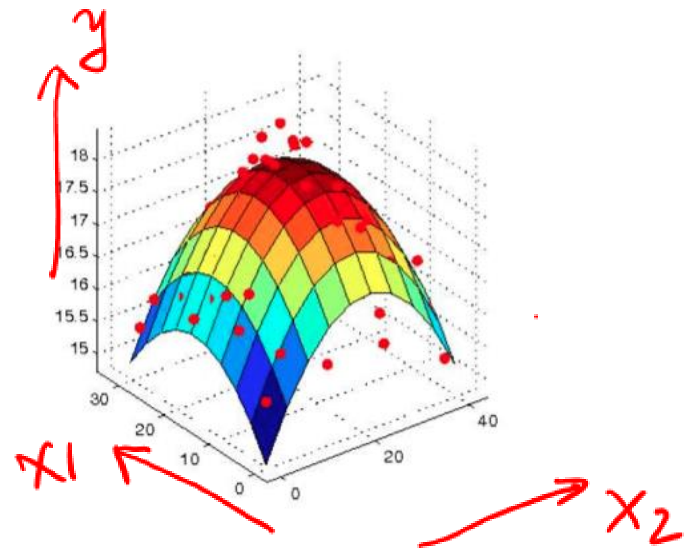
$$\hat{y} = \theta^T \mathbf{x}$$

$$\hat{y} = \theta^T \varphi(\mathbf{x})$$

$$\phi(\mathbf{x}) = [1, x_1, x_2]$$

$$\phi(\mathbf{x}) = [1, x_1, x_2, x_1^2, x_2^2 \, , x_1x_2]$$

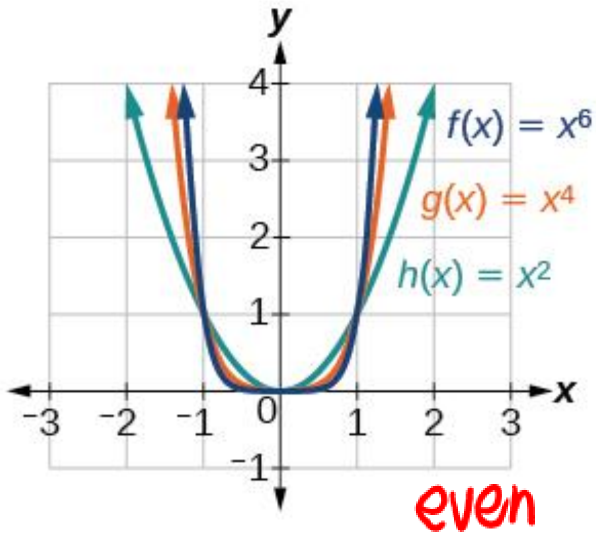


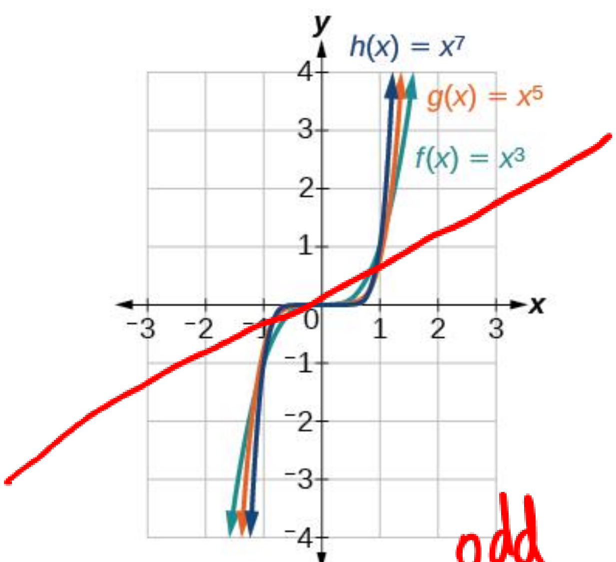


p=2

$$\varphi_j(x) = x^{j-1}$$

Notebook modified version: (I will cell-run it)
https://colab.research.google.com/drive/1gKEk0BuVORnpw_5jQ10wY1ZkA2mSXEAG?usp=sharing





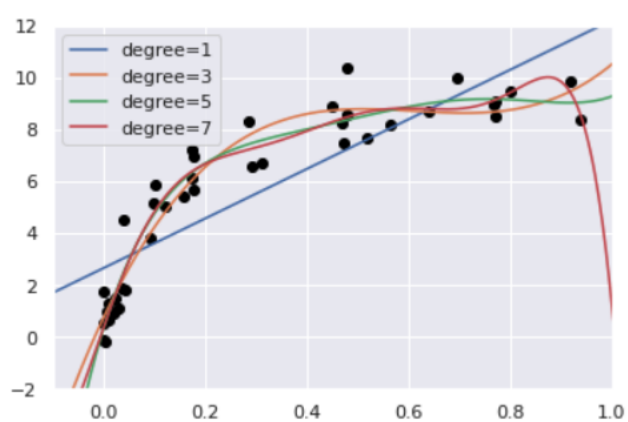
https://colab.research.google.com/github/jakevdp/PythonDataScienceHandbook/blob/master/notebooks/05.06-Linear-Regression.ipynb

My google-colab modified version: (I will cell-run it during class)
https://colab.research.google.com/drive/1gKEk0BuVORnpw_5jQ10wY1ZkA2mSXEAG?usp=sharing

```
%matplotlib inline
import matplotlib.pyplot as plt
import seaborn; seaborn.set()  # plot formatting

X_test = np.linspace(-0.1, 1.1, 500)[:, None]

plt.scatter(X.ravel(), y, color='black')
axis = plt.axis()
for degree in [1, 3, 5, 7]:
    y_test = PolynomialRegression(degree).fit(X, y).predict(X_test)
    plt.plot(X_test.ravel(), y_test, label='degree={0}'.format(degree))
plt.xlim(-0.1, 1.0)
plt.ylim(-2, 12)
plt.legend(loc='best');
```



$[1, x, x^2, \ldots, x^7]$

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Module 2

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$$\hat{y} = \theta^T \mathbf{x} \quad \Rightarrow \quad \hat{y} = \theta^T \varphi(\mathbf{x})$$

Many Possible Basis functions

Recap : Multivariate (non-) Linear Regression with Basis Expansion

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X: Tabular

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$$\hat{y} = \varphi(x)^T \theta$$

Sum of Squared Error (Least Squared)

Normal Equation / GD / SGD

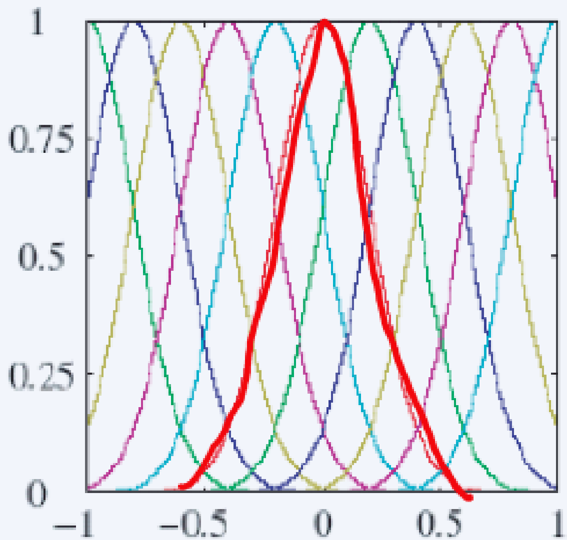
Metrics, Implementation, Regression coefficients w, b

φ: Which and what type?

Many Possible Basis functions

- There are many basis functions, e.g.:
  - Polynomial $\varphi_j(x) = x^{j-1}$ $\left[1, x, x^2, x^3, \ldots, x^d\right]$
  - Radial basis functions $$\varphi_j(x) = \exp\left(-\frac{(x-\mu_j)^2}{2\lambda^2}\right)$$
  - Sigmoidal $$\phi_j(x) = \sigma\left(\frac{x-\mu_j}{s}\right)$$
  - Splines,
  - Fourier,
  - Wavelets, etc

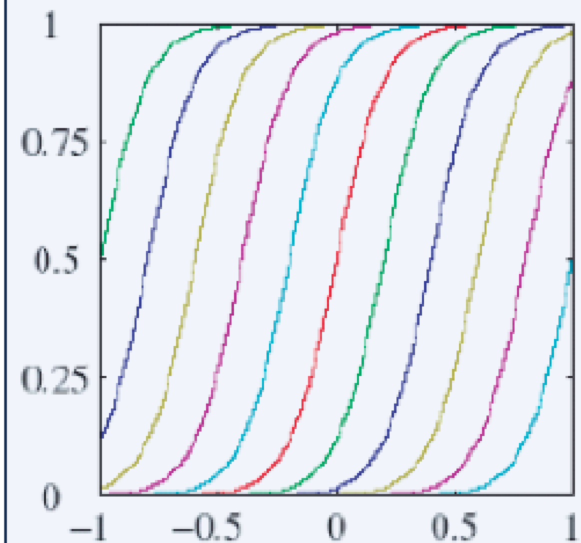
$$\varphi_j(x) = \exp\left(-\frac{(x-\mu_j)^2}{2\lambda^2}\right)$$



Radial basis functions

"bell"-Shaped

$$\phi_j(x) = \sigma\left(\frac{x-\mu_j}{s}\right)$$



Sigmoid functions

"S"-Shaped

RBF = radial-basis function: a function which depends on the radial distance from a Centre point

Gaussian RBF ➔ $$K_\lambda(x,\mu) = \exp\left(-\frac{(x-\mu)^2}{2\lambda^2}\right)$$

as distance from the center increases, the output of the RBF decreases

RBF = radial-basis function: a function which depends on the radial distance from a Centre point

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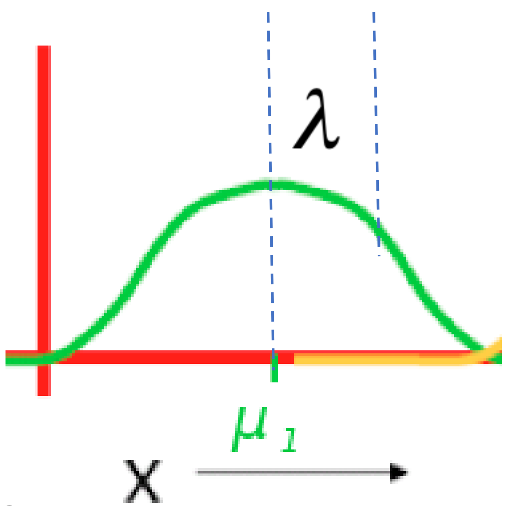
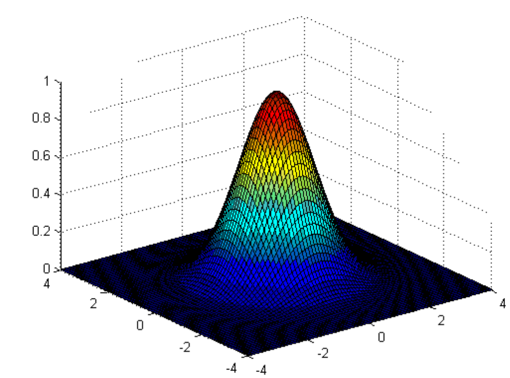
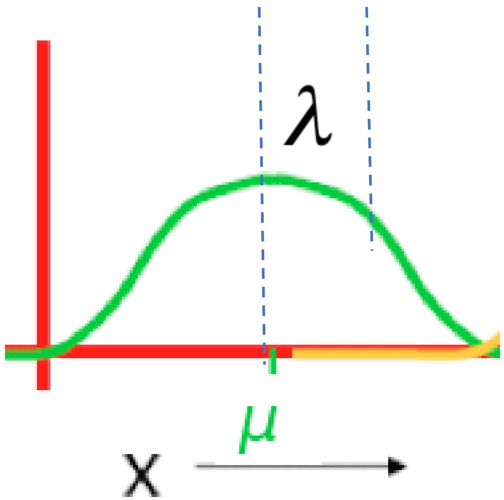


1D case



2D case



$$K_\lambda(x,\mu) = \exp\left(-\frac{(x-\mu)^2}{2\lambda^2}\right)$$

X =	$K_\lambda(x,\mu) =$
μ	1
$\mu+\lambda$	0.6065307
$\mu+2\lambda$	0.1353353
$\mu+3\lambda$	0.0001234098

LR with radial-basis functions

- E.g.: LR with RBF regression:

$$\hat{y} = \theta_0 + \sum_{j=1}^{m} \theta_j \varphi_j(x) = \varphi(x)^T \theta$$

$$\varphi_j(x) := K_{\lambda_j}(x, \mu_j) = \exp\left(-\frac{(x - \mu_j)^2}{2\lambda_j^{\,2}} \right)$$

hyperparameters of RBF basis functions
(the predefined Centers and Width)

LR with radial-basis functions

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$\varphi(x)$: E.g. with four predefined RBF kernels

$$= \left[1, K_{\lambda_1}(x, \mu_1), K_{\lambda 2}(x, \mu_2), K_{\lambda_3}(x, \mu_3), K_{\lambda_4}(x, \mu_4)\right]^T$$

Users need to define the hyperparameters of RBF basis functions (the predefined Centers and Width)

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e.g. (2) LR with radial-basis functions

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$$\vec{\theta} = [\theta_0, \theta_1, \theta_2, \theta_3, \theta_4]^T$$

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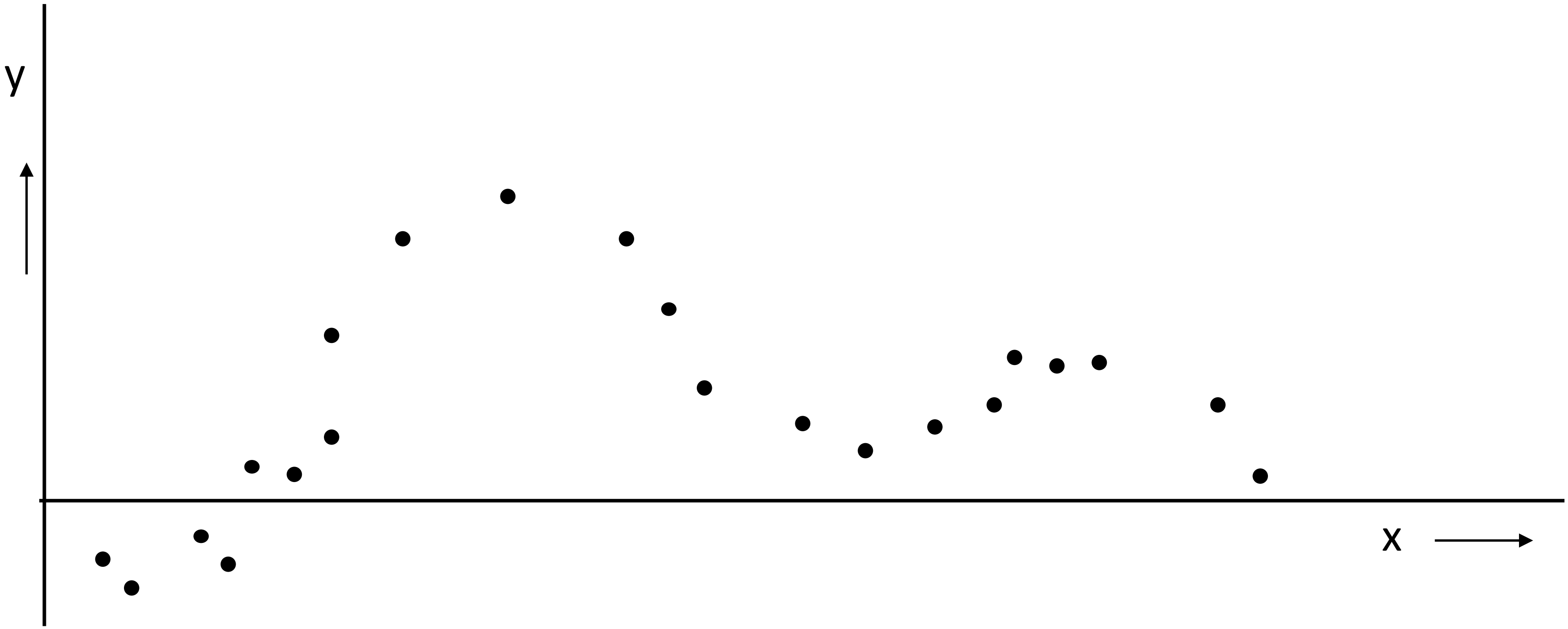
$$\vec{\Theta} = [\Theta_0, \Theta_1, \Theta_2, \Theta_3, \Theta_4]^T$$

$$\left.\begin{array}{cccc} \mu 1 & \mu 2 & \mu 3 & \mu 4 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda 4 \end{array}\right\} \text{hyper para}$$

For example: I want to using 3 RBF basis functions to fit the following data points
(I need to assume 3 predefined centres and width)



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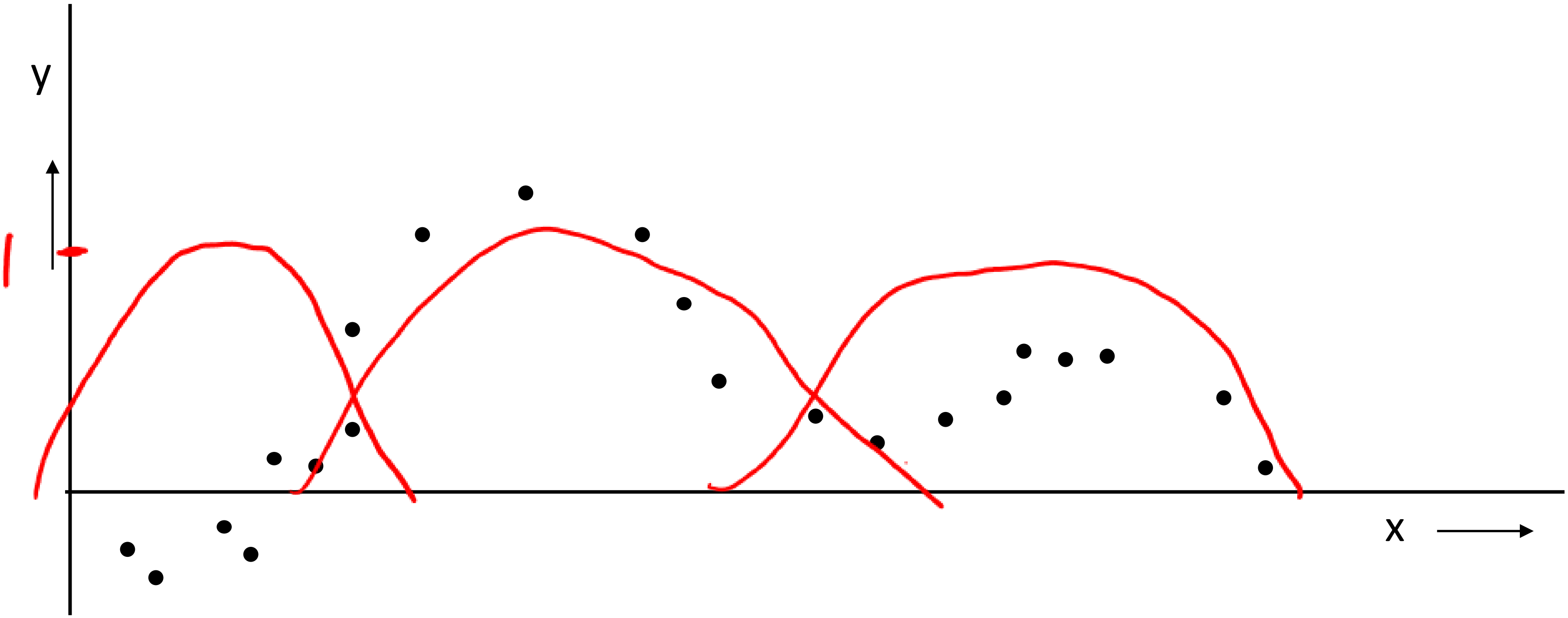


Given Training Data's scatter plot: $(x_i, y_i) \quad i=1,\dots,n$

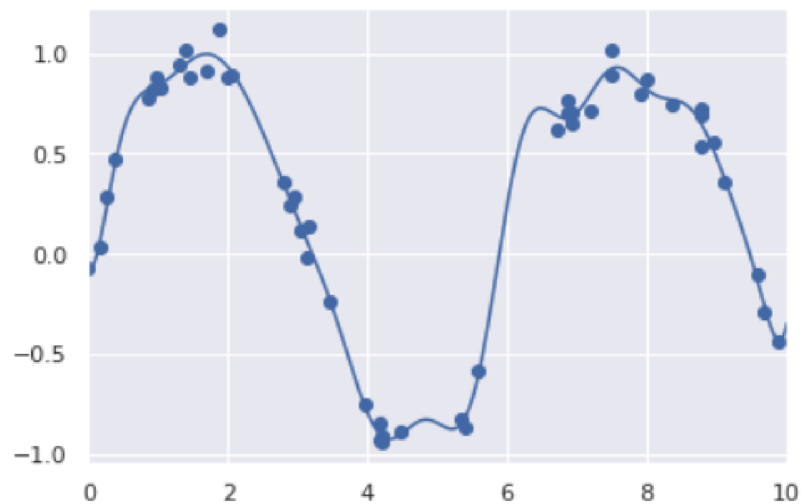
After my 3 RBF fit:

$$f(x) = \theta_0 - 0.5K_{\lambda_1}(x, \mu_1) + 5K_{\lambda_2}(x, \mu_2) + 2K_{\lambda_3}(x, \mu_3)$$

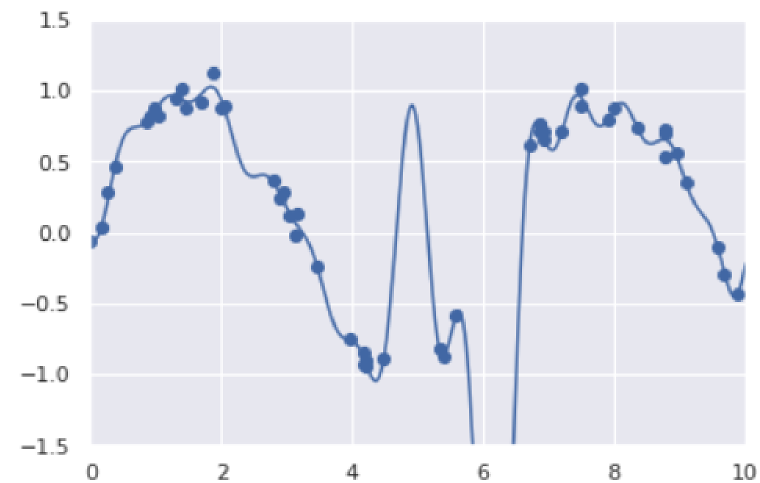
https://colab.research.google.com/drive/1BVcHUBYDO4AlwldcKmpmj5blAbf7lSJb?usp=sharing

Modified from:
https://jakevdp.github.io/PythonDataScienceHandbook/05.06-linear-regression.html

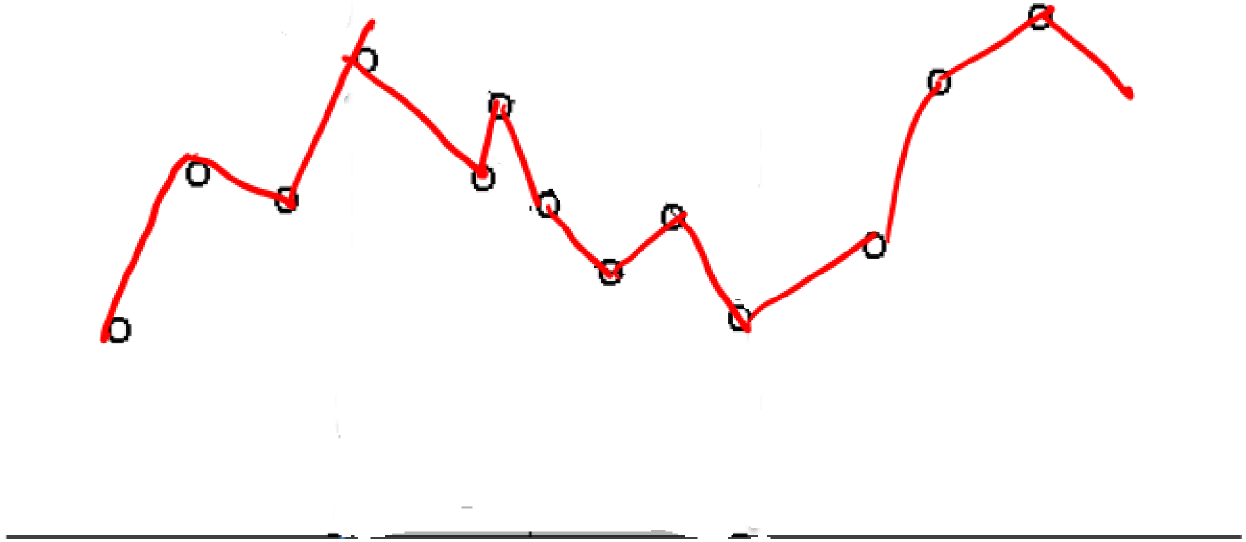
```
gauss_model = make_pipeline(GaussianFeatures(20),
                            LinearRegression())
gauss_model.fit(x[:, np.newaxis], y)
yfit = gauss_model.predict(xfit[:, np.newaxis])

plt.scatter(x, y)
plt.plot(xfit, yfit)
plt.xlim(0, 10);
```




```
model = make_pipeline(GaussianFeatures(30),
                      LinearRegression())
model.fit(x[:, np.newaxis], y)

plt.scatter(x, y)
plt.plot(xfit, model.predict(xfit[:, np.newaxis]))

plt.xlim(0, 10)
plt.ylim(-1.5, 1.5);
```



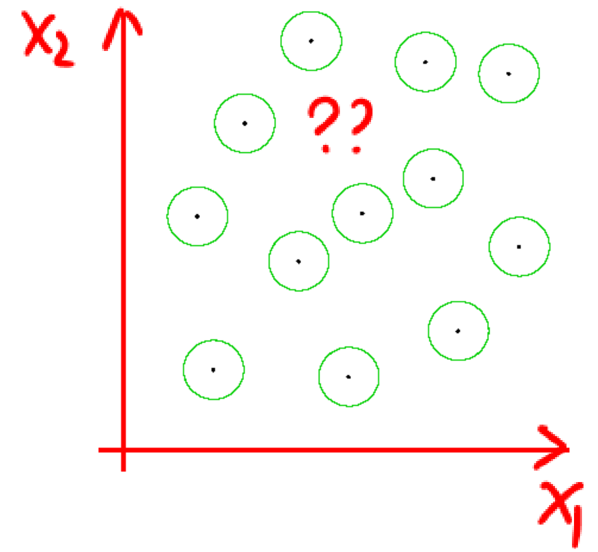
Even more possible Basis Function: RBF, or Piecewise Linear based?



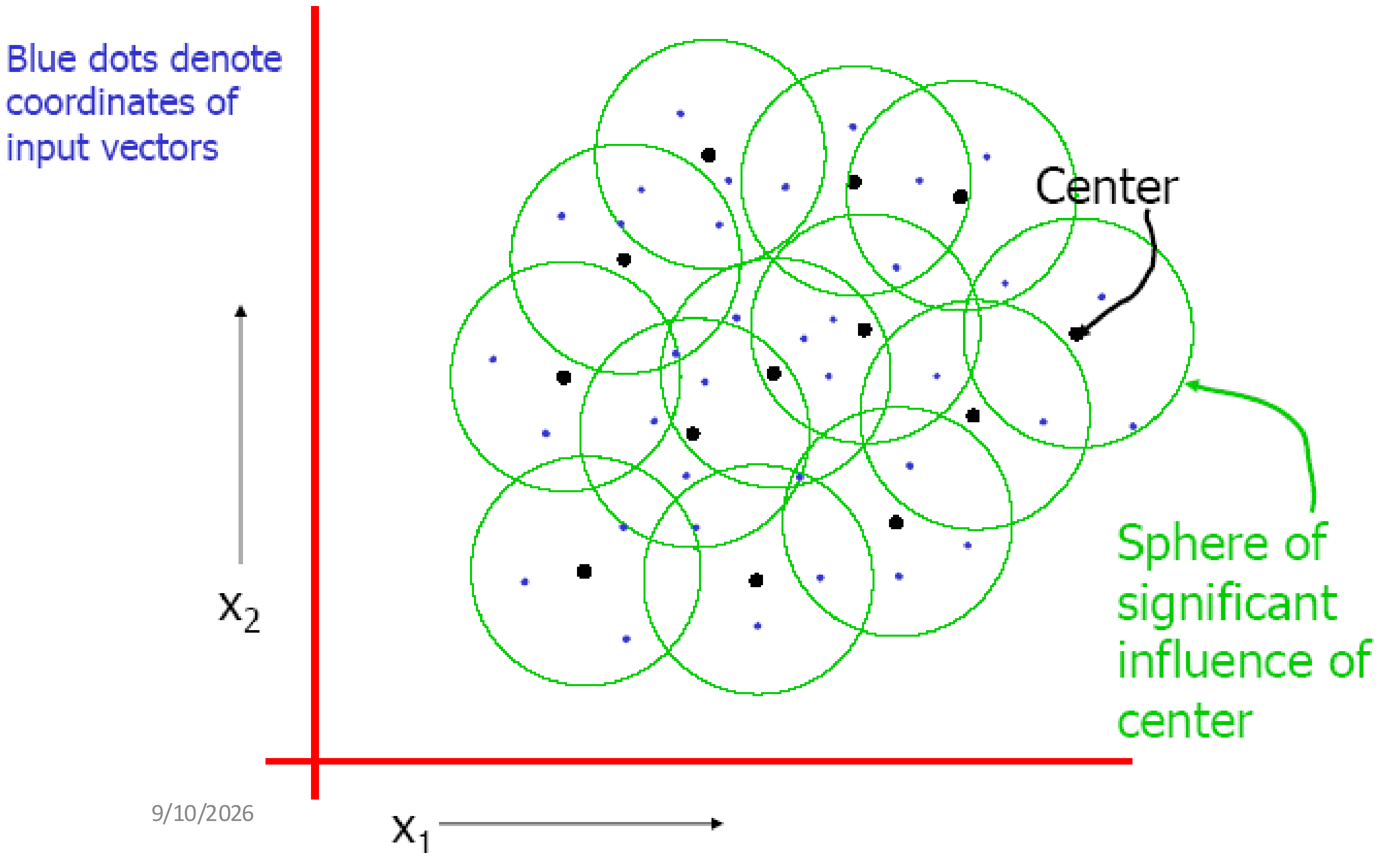
e.g. 2D Good and Bad RBF Basis

No basis can cover
?? points

- A Bad set of predefined 2D RBFs



- A good set of 2D predefined RBF basis



Thank You

Extra

Extra: Nonparametric Regression Models

- K-Nearest Neighbor (KNN) and Locally weighted linear regression are **non-parametric** algorithms.

- The (unweighted) linear regression algorithm that we saw earlier is known as a **parametric** learning algorithm
  - because it has a fixed, finite number of parameters which are fit to the data;
  - Once we've fit the *theta* and stored them away, we no longer need to keep the training data around to make future predictions.
  - In contrast, to make predictions using KNN or locally weighted linear regression, we need to keep the entire training set around.

- The term "non-parametric" (roughly) refers to the fact that the amount of knowledge we need to keep, in order to represent the hypothesis grows with linearly the size of the training set.