UVA CS 4774: Machine Learning

Lecture 5: Linear Regression with Basis Functions Expansion

Dr. Yanjun Qi

University of Virginia

Department of Computer Science

Today : Multivariate (non-) Linear Regression with Basis Expansion

Data: X

↓

Task: y

↓

Representation: : x, f()

↓

Score Function: L()

↓

Search/Optimization : argmin()

↓

Models, Parameters

⇒

X: Tabular

↓

Regression: y continuous

↓

Y = Weighted linear sum of (X basis expansion)

↓

Sum of Squared Error (Least Squared)

↓

Normal Equation / GD / SGD

↓

Metrics, Implementation, Regression coefficients w, b

$$\hat{y} = \theta_0 + \sum_{j=1}^{m} \theta_j \varphi_j(x) = \varphi(x)^T \theta$$

Linear Regression with non-linear basis functions

- LR can deal with nonlinear relationships

$$\hat{y} = \theta^T \mathbf{x} \quad \Longrightarrow \quad \hat{y} = \theta_0 + \sum_{j=1}^{m} \theta_j \varphi_j(\mathbf{x}) = \theta^T \varphi(\mathbf{x})$$

Linear Regression with non-linear basis functions

- **LR can deal with nonlinear relationships**

$$\hat{y} = \theta^T \mathbf{x}$$

$$\hat{y} = \theta_0 + \sum_{j=1}^{m} \theta_j \varphi_j(\mathbf{x}) = \theta^T \varphi(\mathbf{x})$$

$$\vec{\theta} = [\theta_0, \theta_1, \cdots, \theta_m]^T$$

$$\vec{\phi} = [1, \phi_1(x), \cdots, \phi_m(x)]^T$$

LR with non-linear basis functions

- Free to design basis functions (e.g., non-linear features:

Here $\varphi_j(x)$ are predefined basis functions (also $\varphi_0(x) = 1$)

- E.g.: polynomial regression with degree up-to two (d=2) :

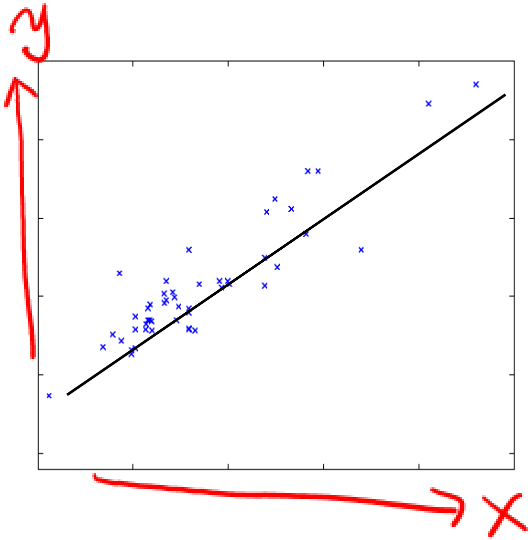
$$\varphi(x) := \left[1, x, x^2\right]^T$$

linear $\vec{x} : [1, x]^T$

Polynomial basis based Regression

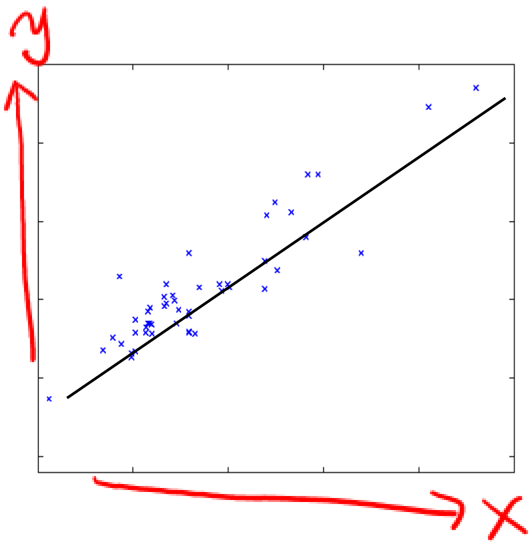
e.g. (1) **polynomial regression**

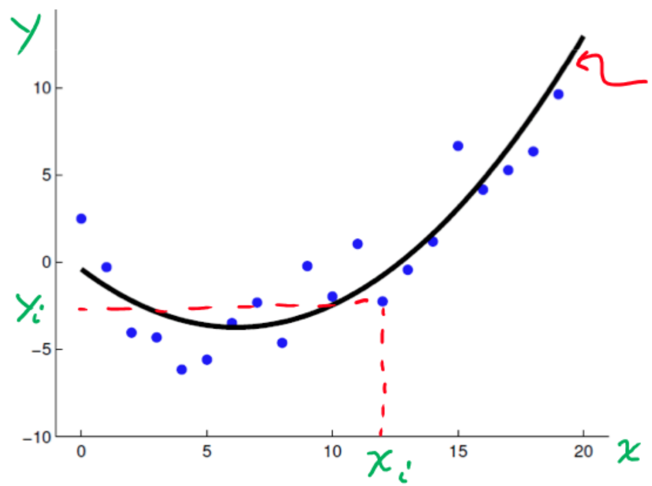
$$\hat{y} = \theta^T \mathbf{x} \quad \Rightarrow \quad \hat{y} = \theta^T \varphi(\mathbf{x})$$



e.g. (1) polynomial regression

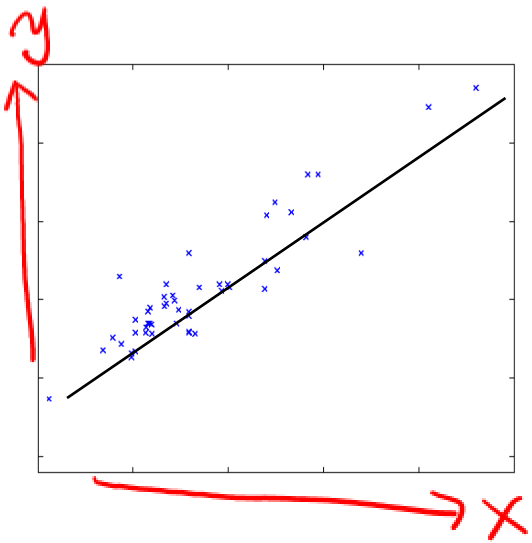
$$\hat{y} = \theta^T \mathbf{x} \quad \Rightarrow \quad \hat{y} = \theta^T \varphi(\mathbf{x})$$





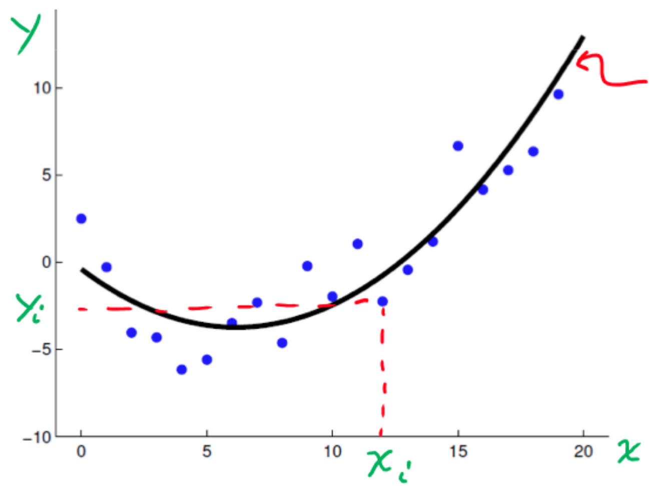
$= \theta_0 + x\theta_1 + x^2\theta_2$

$$\hat{y} = \theta^T \mathbf{x}$$



$$\theta^* = \left(X^T X\right)^{-1} X^T \vec{y}$$

$$\hat{y} = \theta^T \varphi(\mathbf{x})$$



$= \theta_0 + x\theta_1 + x^2\theta_2$

$$\theta^* = \left(\varphi^T \varphi\right)^{-1} \varphi^T \vec{y}$$

$$\varphi(x) := \left[1, x, x^2\right]^T$$

feature engineering

linear $\vec{x} : [1, x]^T$

$$\theta^* = \left(X^T X\right)^{-1} X^T \vec{y}$$

$(p+1)$

$$X_{n\times p} = \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

p=1

$$\varphi(x) := \begin{array}{c} [1, x, x^2]^T \\ 1 \quad 2 \quad \cdots \quad m \end{array}$$

$$\theta^* = \underbrace{\left(\varphi^T \varphi\right)^{-1}}_{m\times n \;\; n\times m} \underbrace{\varphi^T \vec{y}}_{m\times n \;\; n\times 1} \quad m\times 1$$

$$\Phi(x) = \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{array} \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

$1 \cdots m$

$\Downarrow$

$\vec{\theta} \in R^m$

KEY: when the basis func are given, the problem of learning the parameters from data is still LR.

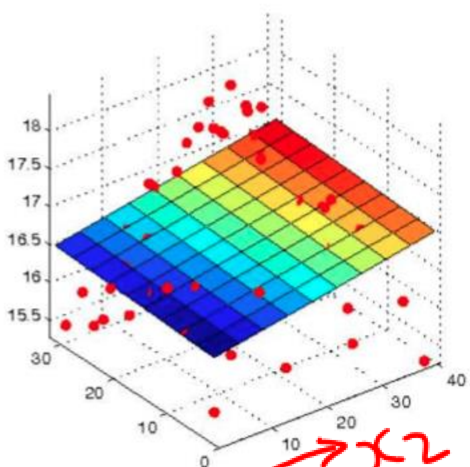
$$\hat{y} = \theta^T \mathbf{x}$$

$$\hat{y} = \theta^T \varphi(\mathbf{x})$$

$$\phi(\mathbf{x}) = [1, x_1, x_2]$$

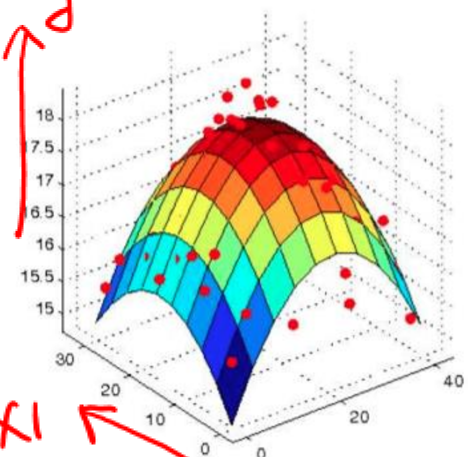
$$\phi(\mathbf{x}) = [1, x_1, x_2, x_1^2, x_2^2] \quad , x_1 x_2$$

y



x_1 x_2

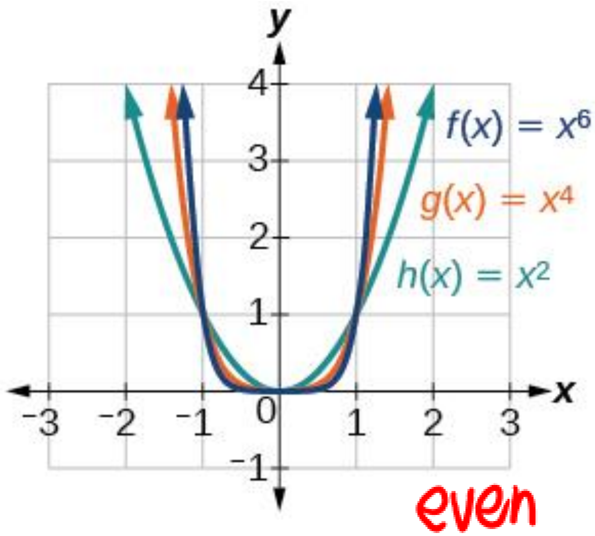
y

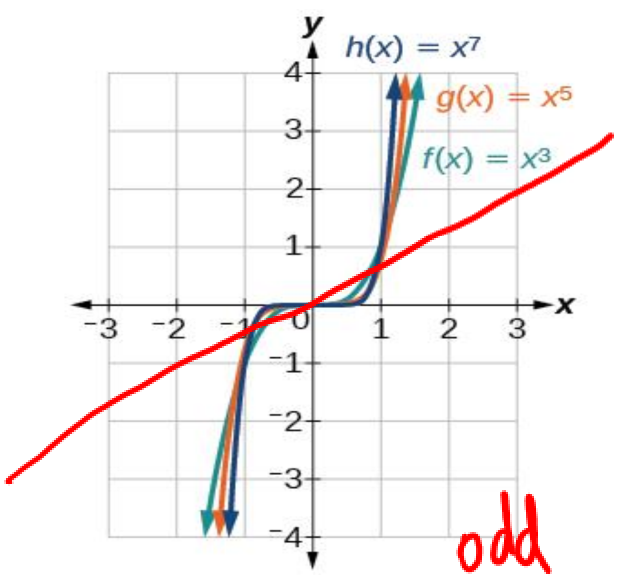


x1 x_2

p=2

$$\varphi_j(x) = x^{j-1}$$





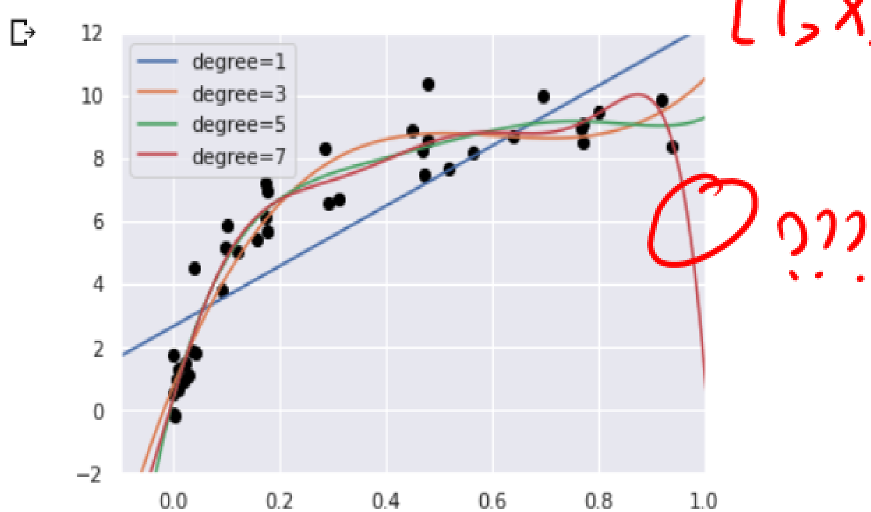
Notebook modified version: (I will cell-run it)
https://colab.research.google.com/drive/1gKEk0BuVORnpw_5jQ10wY1ZkA2mSXEAG?usp=sharing

```
%matplotlib inline
import matplotlib.pyplot as plt
import seaborn; seaborn.set()  # plot formatting

X_test = np.linspace(-0.1, 1.1, 500)[:, None]

plt.scatter(X.ravel(), y, color='black')
axis = plt.axis()
for degree in [1, 3, 5, 7]:
    y_test = PolynomialRegression(degree).fit(X, y).predict(X_test)
    plt.plot(X_test.ravel(), y_test, label='degree={0}'.format(degree))
plt.xlim(-0.1, 1.0)
plt.ylim(-2, 12)
plt.legend(loc='best');
```

$[1, x, x^2, ;, x^7]$



https://colab.research.google.com/github/jakevdp/PythonDataScienceHandbook/blob/master/notebooks/05.06-Linear-Regression.ipynb

My google-colab modified version: (I will cell-run it during class)
https://colab.research.google.com/drive/1gKEk0BuVORnpw_5jQ10wY1ZkA2mSXEAG?usp=sharing

```python
%matplotlib inline
import matplotlib.pyplot as plt
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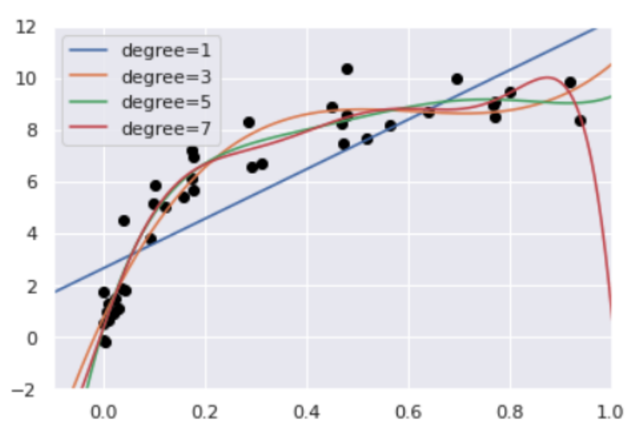

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Lecture 5: Linear Regression with Basis Functions Expansion

Module 2

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$$\hat{y} = \theta^T \mathbf{x} \quad \Rightarrow \quad \hat{y} = \theta^T \varphi(\mathbf{x})$$

Many Possible Basis functions

Recap : Multivariate (non-) Linear Regression with Basis Expansion

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Representation: : x, f()

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Search/Optimization : argmin()

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Y = Weighted linear sum of (X basis expansion)

$$\hat{y} = \varphi(x)^T \theta$$

↓

Sum of Squared Error (Least Squared)

↓

Normal Equation / GD / SGD

↓

Metrics, Implementation, Regression coefficients w, b

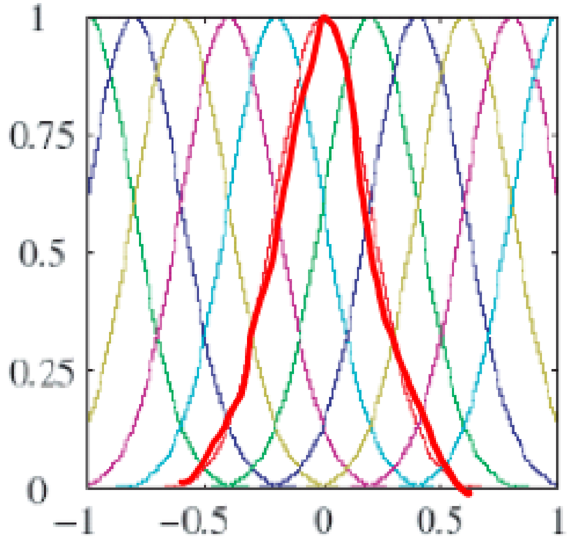
φ: Which and what type?

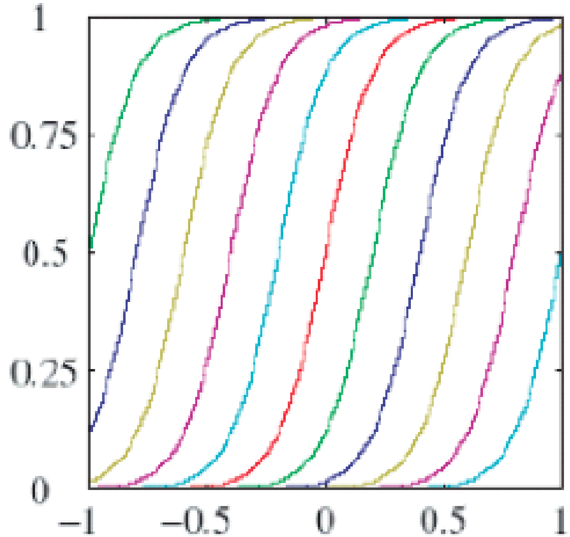
Many Possible Basis functions

- There are many basis functions, e.g.:
  - Polynomial $\varphi_j(x) = x^{j-1}$ $\left[1, x, x^2, x^3, \ldots, x^d\right]$
  - Radial basis functions $$\varphi_j(x) = \exp\left(-\frac{(x-\mu_j)^2}{2\lambda^2}\right)$$
  - Sigmoidal $$\phi_j(x) = \sigma\left(\frac{x-\mu_j}{s}\right)$$
  - Splines,
  - Fourier,
  - Wavelets, etc

$$\varphi_j(x) = \exp\left(-\frac{(x-\mu_j)^2}{2\lambda^2}\right)$$

$$\phi_j(x) = \sigma\left(\frac{x-\mu_j}{s}\right)$$





RBF

Sigmoid

"bell"-Shaped

"S"-Shaped

RBF = radial-basis function: a function which depends on the radial distance from a Centre point

Gaussian RBF ➔

$$K_{\lambda}(x,r) = \exp\left(-\frac{(x-r)^2}{2\lambda^2}\right)$$

as distance from the center r increases, the output of the RBF decreases

RBF = radial-basis function: a function which depends on the radial distance from a Centre point

Gaussian RBF → $$K_\lambda(x,r) = \exp\left(-\frac{(x-r)^2}{2\lambda^2}\right)$$

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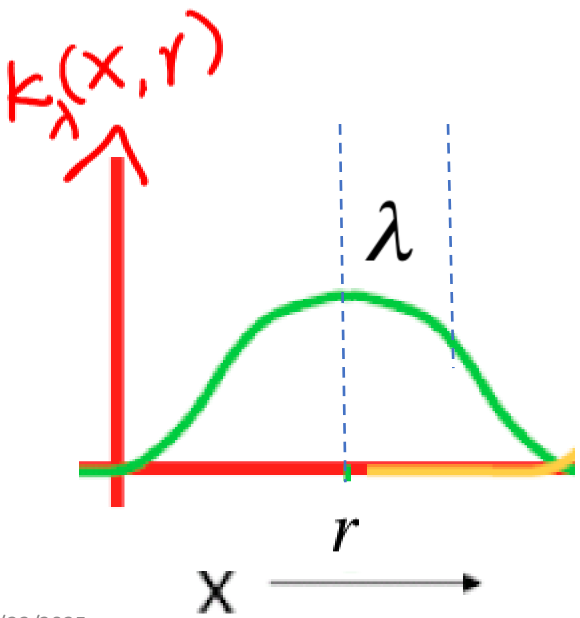
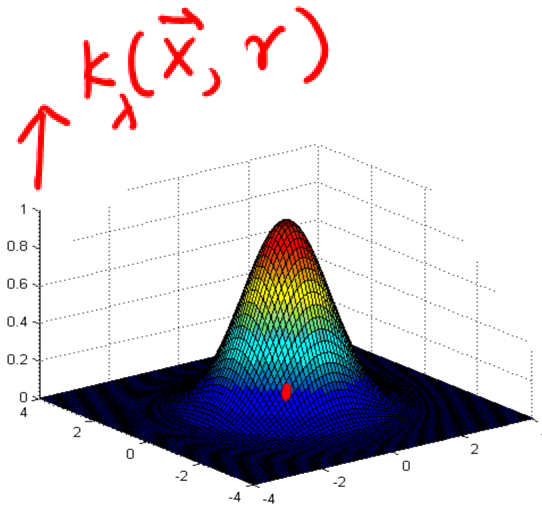
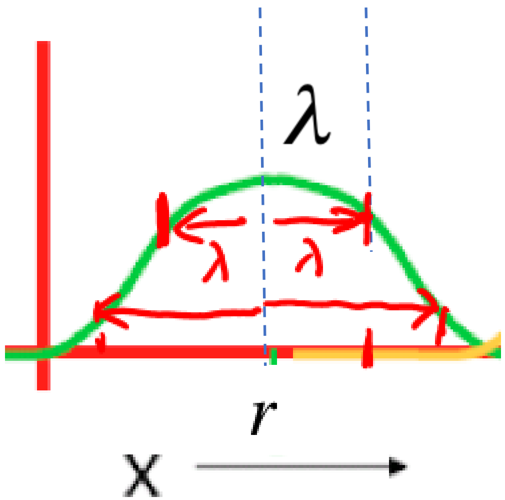


1D case



2D case



$$K_\lambda(x,r) = \exp\left(-\frac{(x-r)^2}{2\lambda^2}\right)$$

r

$X =$	$K_\lambda(x,r) =$
r	1
$r+\lambda$	0.6065307
$r+2\lambda$	0.1353353
$r+3\lambda$	0.0001234098 ~ 0

θ

$$\begin{cases} x > r+3\lambda \\ x < r-3\lambda \end{cases}$$

LR with radial-basis functions

- E.g.: LR with RBF regression:

$$\hat{y} = \theta_0 + \sum_{j=1}^{m} \theta_j \varphi_j(x) = \varphi(x)^T \theta$$

$$\varphi_j(x) := K_{\lambda_j}(x, r_j) = \exp\left(-\frac{(x - r_j)^2}{2{\lambda_j}^2}\right)$$

hyperparameters of RBF basis functions
(the predefined Centers and Width)

LR with radial-basis functions

- E.g.: LR with RBF regression:

$$\hat{y} = \theta_0 + \sum_{j=1}^{m} \theta_j \varphi_j(x) = \varphi(x)^T \theta$$

$$\varphi_j(x) := K_{\lambda_j}(x, r_j) = \exp\left(-\frac{(x - \mathrm{r}_j)^2}{2\lambda_j{}^2}\right)$$

$\varphi(x)$: E.g. with four predefined RBF kernels

$$= \left[1, K_{\lambda_1}(x, r_1), K_{\lambda 2}(x, r_2), K_{\lambda_3}(x, r_3), K_{\lambda_4}(x, r_4)\right]^T$$

Users need to define the hyperparameters of RBF basis functions (the predefined Centers and Width)

$\varphi(x)$: E.g. with four predefined RBF kernels

$$= \left[1, K_{\lambda_1}(x, r_1), K_{\lambda 2}(x, r_2), K_{\lambda_3}(x, r_3), K_{\lambda_4}(x, r_4)\right]^T$$

$$\theta^* = \left(\varphi^T \varphi\right)^{-1} \varphi^T \vec{y}$$

e.g. (2) LR with radial-basis functions

- E.g.: LR with RBF regression:

$$\hat{y} = \theta_0 + \square_{j=1}^{m} \theta_j \varphi_j(x) = \varphi(x)^T \theta$$

$$\varphi(x) := \left[1, K_{\lambda_1}(x, r_1), K_{\lambda 2}(x, r_2), K_{\lambda_3}(x, r_3), K_{\lambda 4}(x, r_4)\right]^T$$

$$\vec{\theta} = [\theta_0, \theta_1, \theta_2, \theta_3, \theta_4]^T$$

$$\theta^* = \left(\varphi^T \varphi\right)^{-1} \varphi^T \vec{y}$$

e.g. (2) LR with radial-basis functions

- E.g.: LR with RBF regression:

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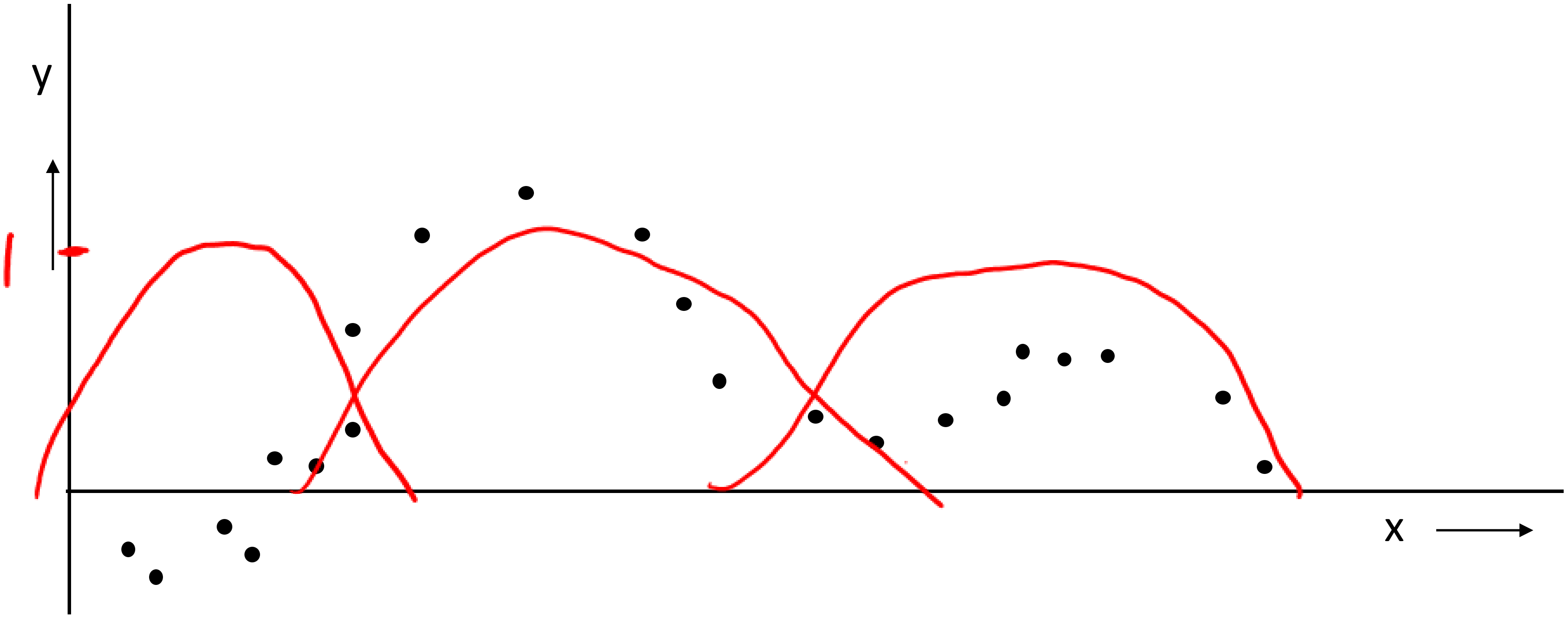
$$\theta^* = \left(\varphi^T \varphi\right)^{-1} \varphi^T \vec{y}$$

$$\vec{\theta} = [\theta_0, \theta_1, \theta_2, \theta_3, \theta_4]^T$$

$$\begin{array}{cccc} r_1 & r_2 & \lambda_3 & r_4 \\ \lambda_1 & \lambda_2 & r_3 & \lambda 4 \end{array} \Bigg\} \text{hyper para}$$

For example: I want to using 3 RBF basis functions to fit the following data points
(I need to assume 3 predefined centres and width)



Given Training Data's scatter plot: $(x_i, y_i) \quad i=1,\dots,n$

After my 3 RBF fit:

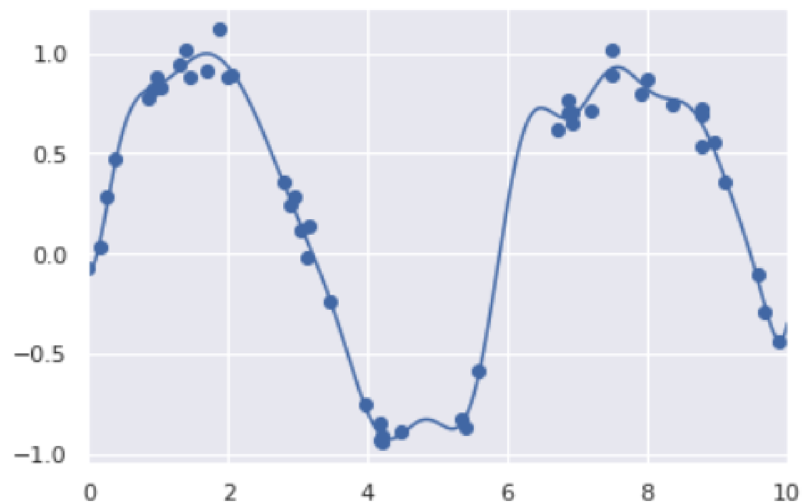
$$f(x) = \theta_0 - 0.5\, K_{\lambda_1}(x, r_1) + 5\, K_{\lambda_2}(x, r_2) + 2\, K_{\lambda_3}(x, r_3)$$

https://colab.research.google.com/drive/1BVcHUBYDO4AlwldcKmpmj5blAbf7lSJb?usp=sharing

Modified from:
https://jakevdp.github.io/PythonDataScienceHandbook/05.06-linear-regression.html

```
gauss_model = make_pipeline(GaussianFeatures(20),
                            LinearRegression())
gauss_model.fit(x[:, np.newaxis], y)
yfit = gauss_model.predict(xfit[:, np.newaxis])

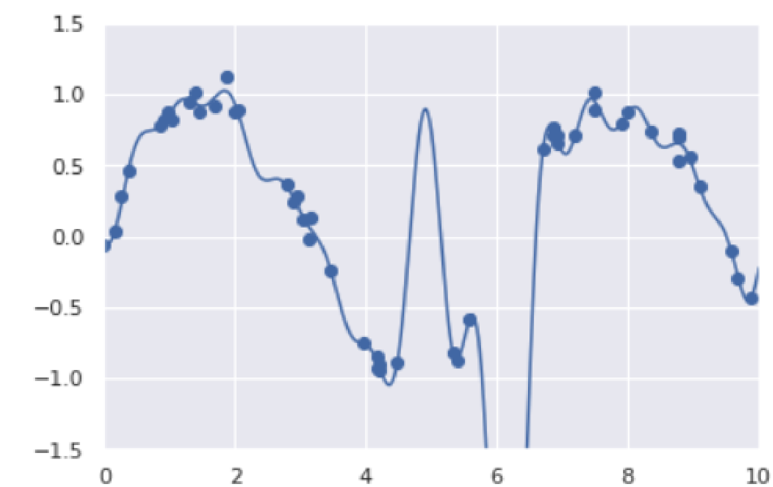
plt.scatter(x, y)
plt.plot(xfit, yfit)
plt.xlim(0, 10);
```



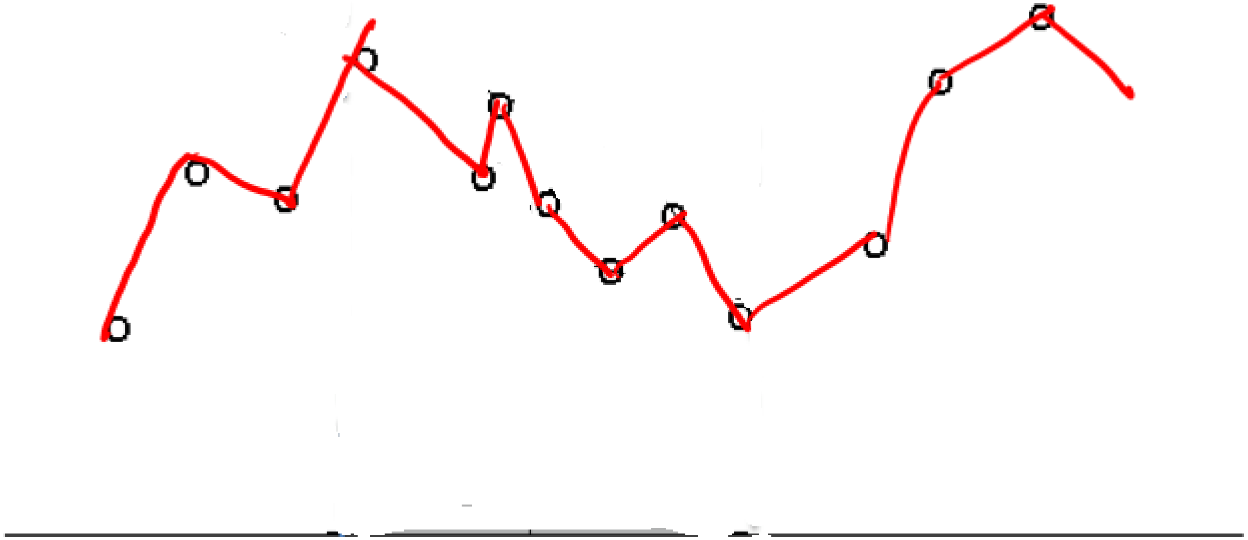

```
model = make_pipeline(GaussianFeatures(30),
                      LinearRegression())
model.fit(x[:, np.newaxis], y)

plt.scatter(x, y)
plt.plot(xfit, model.predict(xfit[:, np.newaxis]))

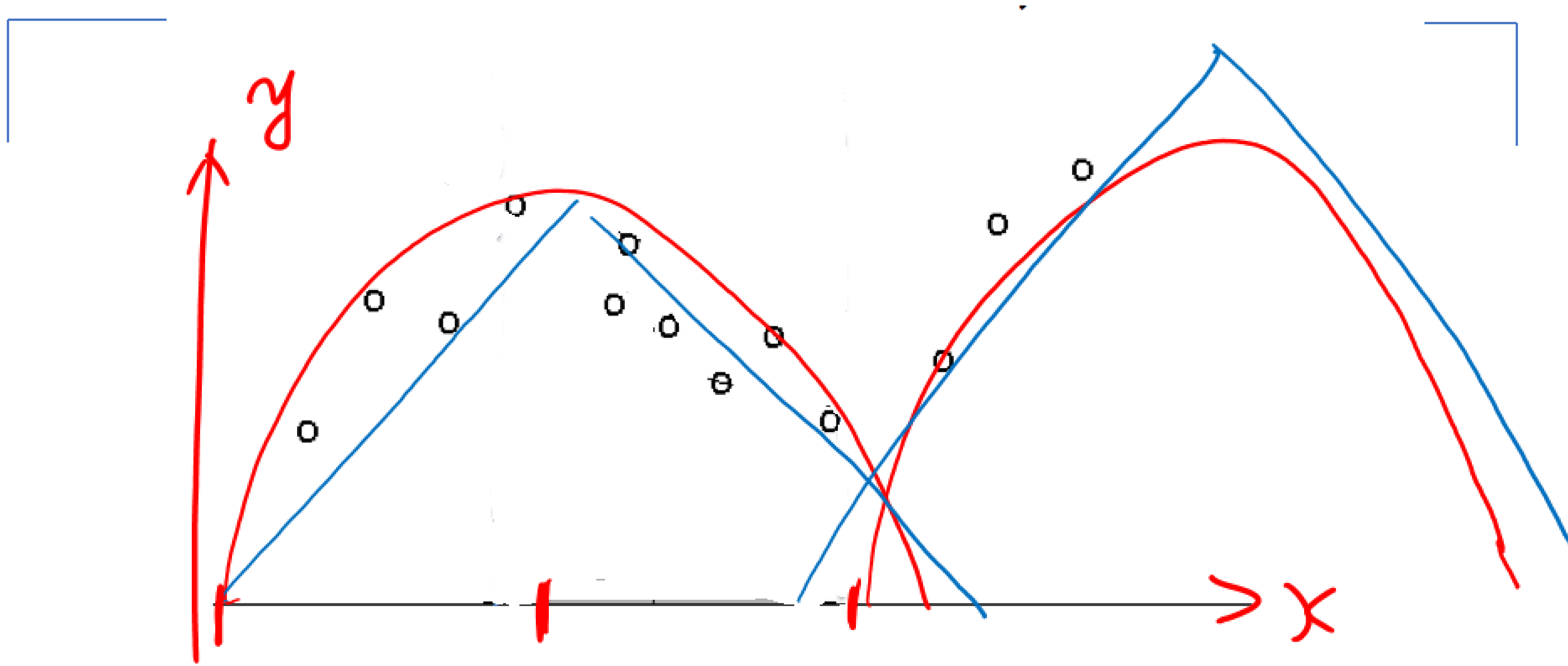
plt.xlim(0, 10)
plt.ylim(-1.5, 1.5);
```



Extra: even more possible Basis Function: RBF, or Piecewise Linear based?



e.g. Even more possible Basis Func?



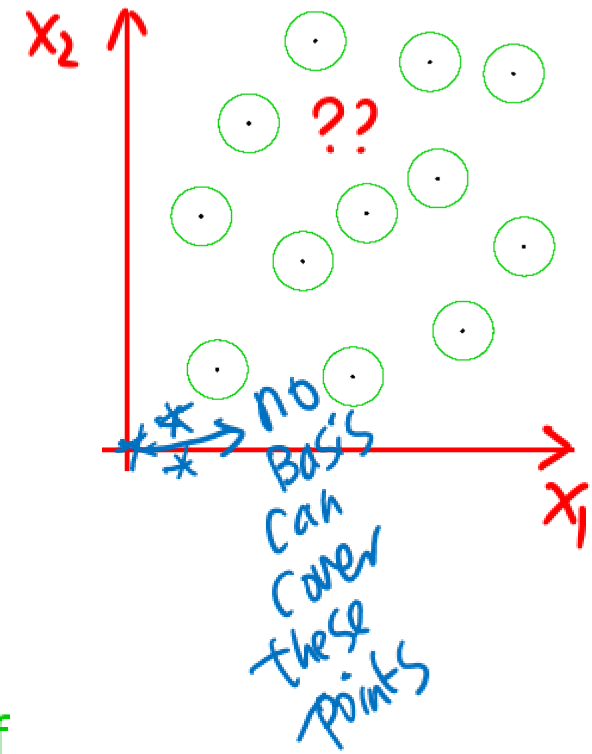
Thank You

Extra

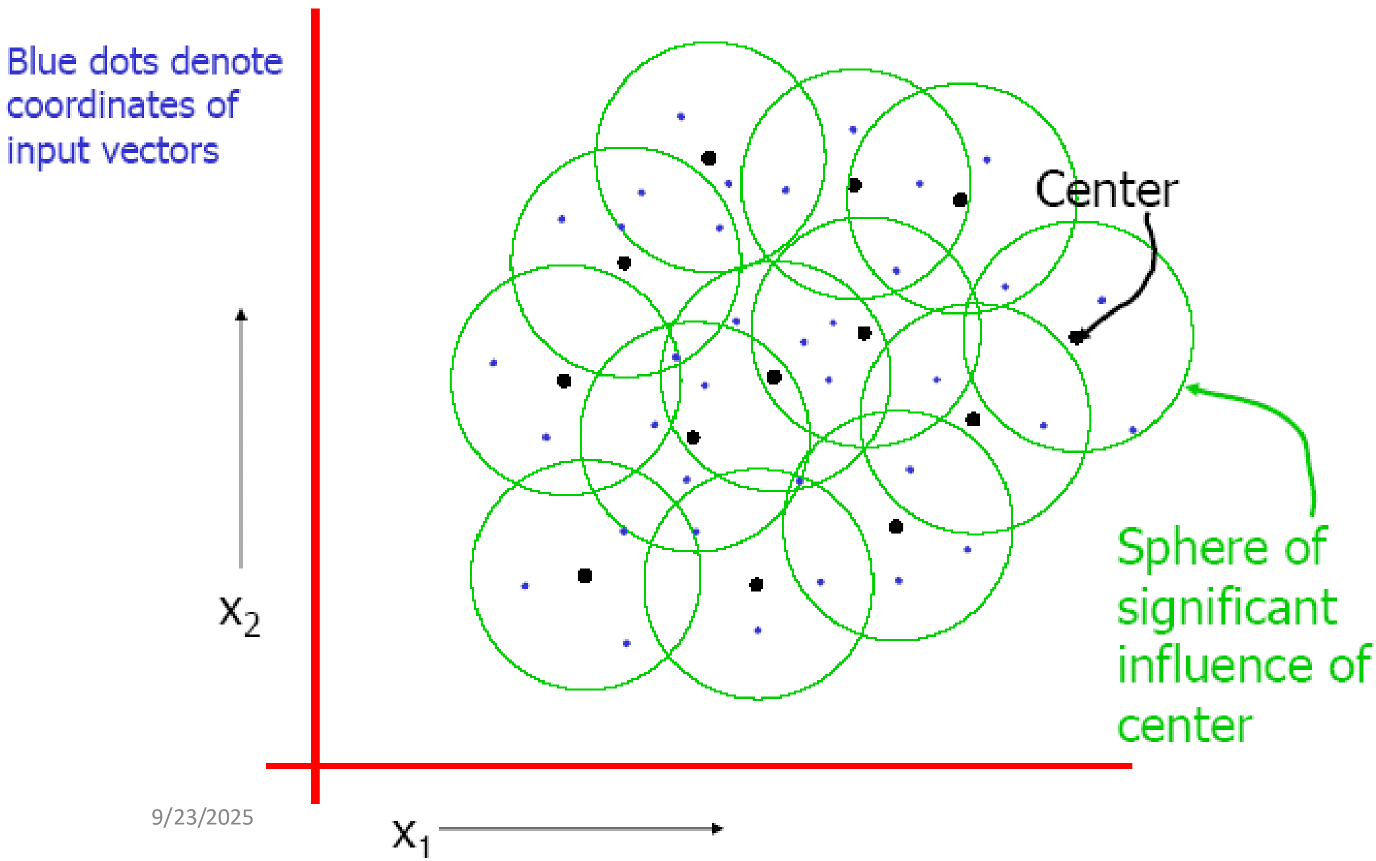
Extra: e.g. 2D Good and Bad RBF Basis

Contour view

- A Bad set of predefined 2D RBFs



- A good set of 2D predefined RBF basis



Extra: Nonparametric Regression Models

- K-Nearest Neighbor (KNN) and Locally weighted linear regression are **non-parametric** algorithms.

- The (unweighted) linear regression algorithm that we saw earlier is known as a **parametric** learning algorithm
  - because it has a fixed, finite number of parameters which are fit to the data;
  - Once we've fit the *theta* and stored them away, we no longer need to keep the training data around to make future predictions.
  - In contrast, to make predictions using KNN or locally weighted linear regression, we need to keep the entire training set around.

- The term "non-parametric" (roughly) refers to the fact that the amount of knowledge we need to keep, in order to represent the hypothesis grows with linearly the size of the training set.